EQUILIBRIUM MEASURES FOR CERTAIN ISOMETRIC EXTENSIONS OF ANOSOV SYSTEMS

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ABSTRACT. We prove that for the frame flow on a negatively curved, closed manifold of odd dimension other than 7, and a Hölder continuous potential that is constant on fibers, there is a unique equilibrium measure. Brin and Gromov's theorem on the ergodicity of frame flows follows as a corollary. Our methods also give a corresponding result for automorphisms of the Heisenberg manifold fibering over the torus.

1. INTRODUCTION

Topological entropy is a measure of the complexity of a dynamical system on a compact topological space. It records the exponential growth rate of the amount of information needed to capture the system for time t at a fine resolution ϵ, as $t \to \infty$ and $\epsilon \to 0$. As a topological invariant, it can be used to distinguish between topologically different dynamical systems. Positive topological entropy is also used as an indicator of chaos. One may wish to be selective of the information in the system to include, however; such a selection can be encoded by a probability measure. Given an invariant probability measure, *measure theoretic entropy* computes the complexity of the dynamical system as seen by the measure. This, of course, depends on the measure chosen. These two ways of computing entropy are related by the *variational principle*, which states that the topological entropy is the supremum of measure-theoretic entropies over the set of f-invariant measures. The variational principle provides a tool for picking out distinguished measures—namely, those that maximize the measure-theoretic entropy. Such measures (if they exist) are called *measures of maximal entropy.*

Pressure is a generalization of entropy which takes into account a weighting of the contribution of each orbit to the entropy by a Hölder continuous potential function. In the case that the potential is identically zero, the pressure is just the entropy of the system. The variational principle also applies to topological and measure theoretic pressure and implies that for a given dynamical system, any Hölder continuous potential function determines a set (possibly empty) of invariant measures that maximize the measure theoretic pressure. Such measures are called *equilibrium measures.* As is well known, work of Newhouse and Yomdin shows that equilibrium measures for continuous potential functions and C^∞ dynamics always exist [26].

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In the 1970's, Bowen and Ruelle produced a set of results considering entropy, pressure, and equilibrium measures for Axiom A and, in particular, Anosov diffeomorphisms and flows [4, 5]. A central result is the following: given a transitive Anosov diffeomorphism or flow of a compact manifold and a Hölder continuous potential function, there exists an equilibrium measure and it is unique. Moreover, this measure is ergodic and has full support. While this theorem applies broadly to Anosov dynamics, there is no general theory for partially hyperbolic systems. Results are limited to specific sets of diffeomorphisms and potentials (quite often only the zero potential). In the main part of this paper, we study equilibrium measures for the (full) frame flow F^t on a negatively curved, closed manifold M and a particular class of potentials. Recall that F^t is a flow on the positively oriented orthonormal frame bundle FM, which factors over the unit tangent bundle SM (see Section 2 for definitions). When the dimension of the underlying manifold M is at least 3, such frame flows are (non-Anosov) partially hyperbolic flows. Indeed, the orthonormal frame bundle fibers non-trivially over the unit tangent bundle if $\dim(M) \geq 3$, and the frame flow is isometric along the fibers. This isometric behavior along a foliation plays an important role in proving the following result.

Theorem 1. *Let M be a closed, oriented, negatively curved n-manifold, with n odd and not equal to 7. For any Hölder continuous potential $\varphi : FM \to \mathbb{R}$ that is constant on the fibers of the bundle $FM \to SM$, there is a unique equilibrium measure for (F^t, φ). It is ergodic and has full support.*

Let us make a couple of comments on the assumptions of this theorem. First, the restriction on the dimension of the manifold in this theorem is due to a topological argument using the non-existence of certain transitive actions on spheres. Second, while the condition that the potential function is constant on the fibers is highly restrictive, it does apply to any Hölder function pulled back from a function on the unit tangent bundle. In particular, the theorem applies to the constant potentials, whose equilibrium measure is the measure of maximal entropy. It also applies to the unstable Jacobian potential, whose equilibrium measure for the geodesic flow is Liouville measure. This assumption on the potential makes the measure amenable to the methods used in the proof (namely, the projected measure has local product structure). We believe that the theorem should hold for a more general class of functions, but the problem becomes much more difficult.

The methods of the proof also apply in other situations, for instance to certain automorphisms of a nilmanifolds that factor over an Anosov map. Here is one such example; we hope to pursue these matters in greater detail in the future. Let $Heis$ be the 3-dimensional Heisenberg group, and $Heis(\mathbb{Z})$ its integer lattice. Let $M = Heis/Heis(\mathbb{Z})$ be the Heisenberg manifold. Note that M naturally fibers over the 2-torus $\mathbb{T}^2$ by factoring by the center of $Heis$.

Theorem 2. *Let M be the Heisenberg manifold, and f a partially hyperbolic automorphism of M such that the induced action on the base torus is Anosov and the action on the fibers is isometric. Then for any Hölder continuous potential $\varphi : M \to \mathbb{R}$ that is constant on the fibers of the canonical projection map $M \to \mathbb{T}^2$, there is a unique equilibrium measure for (f, φ). It is ergodic and has full support.*

We remark that the equilibrium states of Theorems 1 or 2 are in one-to-one correspondence with cohomology classes in the set of potential functions constant on fibers. Indeed, the equilibrium measure is uniquely determined by an equilibrium measure for the Anosov base dynamics, where this is a classical result.

Recent study of the existence and uniqueness of equilibrium measures for partially hyperbolic diffeomorphisms and flows focuses on examples. The following is a list of results most pertinent to the present paper. For partially hyperbolic automorphisms of tori, it is a classical result of Berg that Haar measure is the unique measure of maximal entropy [2, 14]. The existence and uniqueness of a measure of maximal entropy for diffeomorphisms of the 3-torus homotopic to a hyperbolic automorphism was shown by Ures in [29]. Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi, and Ures proved existence and uniqueness of the measure of maximal entropy for 3-dimensional partially hyperbolic diffeomorphisms with compact center leaves when the central Lyapunov exponent is zero, and multiple measures of maximal entropy when the central Lyapunov exponent is non-zero [28]. Climenhaga, Fisher, and Thompson showed existence and uniqueness of equilibrium measures under conditions on the potential function for certain derived-from-Anosov diffeomorphisms of tori [10]. Finally, Knieper proved for geodesic flows in higher rank symmetric spaces that the measure of maximal entropy is again unique [20] with support a submanifold of the unit tangent bundle on which the geodesic flow is partially hyperbolic. We note that he also proved uniqueness of the measure of maximal entropy for the geodesic flow on closed rank 1 manifolds of nonpositive curvature [19]. This was recently generalized by Burns, Climenhaga, Fisher, and Thompson to equilibrium states for potential functions satisfying a bounded range hypothesis [8]. These flows are non-uniformly hyperbolic but not usually partially hyperbolic.

Bowen and Ruelle studied equilibrium measures for uniformly hyperbolic diffeomorphisms and flows via expansivity and specification. These results have been extended to weak versions of expansivity and specification by Climenhaga and Thompson in [11], and used by them and Fisher in [10]. To our knowledge, outside of measures of maximal entropy, Theorems 1 and 2 are the first results about the uniqueness of equilibrium measures for partially hyperbolic systems which have no regions of uniform hyperbolicity.

The main method of proof is to combine ideas from measure rigidity of higher rank abelian actions with ideas from the proof of Livšic' theorem on measurable cohomology of Hölder cocycles [25]. More precisely, we disintegrate an equilibrium state into conditional measures along the central foliation. The support of a conditional measure generates limits of isometries along central leaves which act transitively on this support. Thus, for frame flows, one arrives at a dichotomy for conditional measures: they are either invariant under the action of $SO(n-1)$ and hence a scalar multiple of Haar measure or else they are invariant under a proper subgroup of $SO(n-1)$. In the first case, we reduce the problem to understanding the equilibrium state projected to the unit tangent bundle. There, the projected flow is Anosov and classical methods apply to prove uniqueness and ergodicity of the projected measure. This in turn implies that the conditional measures are a constant multiple of Haar measure, as desired. In the second case, we get an invariant measurable section of an associated bundle. As discussed in [16], the ideas of Livšic then show that such sections have to be continuous and even smooth, giving us a reduction of structure group of the frame bundle. In the case that n is odd and not 7, this is a contradiction, as shown by Brin and Gromov. Similar considerations and topological restrictions apply in the case of the Heisenberg manifold in Theorem 2.

The idea to study invariant measures via their conditional measures along isometric foliations was introduced in [18], and used repeatedly in other works (e.g., [1, 12, 13, 23]). In particular, Lindenstrauss and Schmidt analyzed invariant measures for partially hyperbolic automorphisms of tori and more general compact abelian groups in [23, 24]. They showed that for ergodic measures singular with respect to Haar measure, the conditional measures along central foliations must be finite. We note, however, that there are many such measures. Indeed, every number in the interval $[0, h_{top}(f)]$ is the measure theoretic entropy for some invariant measure, by the universality theorem of Quas and Soo for automorphisms of tori [27]. Equilibrium measures are of course much more special. While the situation is classical and well understood for hyperbolic toral automorphisms, the non-expansive case is unclear: are equilibrium states unique for a given potential function? How many equilibrium states are there in total? Measure rigidity techniques only give limited information. In contrast with general invariant measures for partially hyperbolic toral automorphisms, we find for the Heisenberg manifold and a specific family of potential functions that equilibrium measures are unique and are uniquely determined by the cohomology class of the potential function. We also remark that Avila, Viana, and Wilkinson studied conditional measures on center leaves and their invariance under stable and unstable holonomy in their work on measure rigidity of perturbations of the time 1 map of the geodesic flow of a hyperbolic surface [1].

Finally, the following classical result by Brin and Gromov on the ergodicity of certain frame flows follows as a corollary to Theorem 1.

Theorem 3 (Brin-Gromov, [6])**.** *Let M be an odd dimensional closed, oriented manifold of negative sectional curvature and dimension $n \neq 7$. Then the frame flow is ergodic.*

We remark that Brin and Karcher [7] proved ergodicity of the frame flow in even dimensions $\neq 8$ under pinching assumptions on the curvature. These results were extended under pinching restrictions on the curvature to dimensions 7 and 8 by Burns and Pollicott in [9]. Our approach does not apply to such results, since those authors use pinching to control Brin-Pesin groups.

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2. Preliminaries

We first review some basic definitions and results.

2.1. **Frame flow.** Let M be a closed, oriented n-dimensional manifold with Riemannian metric. Let $SM = \{(x, v) : x \in M, v \in T_xM, \|v\| = 1\}$ denote the unit tangent bundle, and let $FM = \{(x; v_0, v_1, \dots, v_{n-1}) : x \in M, v_i \in T_xM\}$, where the v_i form a positively oriented orthonormal frame at x, be the frame bundle. The metric induces a geodesic flow $g^t : SM \to SM$, defined by $g^t(x, v) =$

$(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$, where $\gamma_{(x,v)}$ is the unique geodesic determined by the vector (x, v). The metric also induces a frame flow $F^t : FM \to FM$; defined by

$$F^t(x, v_0, v_1, \ldots, v_{n-1}) = (g^t(x, v_0), \Gamma^t_\gamma(v_1), \ldots, \Gamma^t_\gamma(v_{n-1})),$$

where Γ^t_γ denotes parallel transport along the geodesic $\gamma_{(x,v_0)}$. The frame bundle is a fiber bundle over SM, with structure group $SO(n-1)$ acting on the frames by rotations that keep the vector v_0 fixed. Hence, we have the following commuting diagram:

$$\begin{array}{ccc} FM & \xrightarrow{F^t} & FM \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ SM & \xrightarrow{g^t} & SM \end{array}$$

The frame flow preserves a natural smooth measure $\mu = \mu_L \times \lambda_{SO(n-1)}$, where μ_L is (normalized) Liouville measure on the unit tangent bundle, and $\lambda_{SO(n-1)}$ is Haar measure on $SO(n-1)$. Note that $\pi_*\mu = \mu_L$, and μ_L is preserved by the geodesic flow.

2.2. **Partial hyperbolicity.** A flow $f^t : X \to X$ on a manifold X with a Riemannian metric is called *partially hyperbolic* if the tangent bundle splits into three subbundles $TX = E^s \oplus E^c \oplus E^u$, each invariant under the flow, such that vectors in E^s are (eventually) exponentially contracted by the flow, the vectors in E^u are (eventually) exponentially expanded by the flow, and any contraction (resp. expansion) of vectors in E^c is dominated by that of vectors in E^s (resp. E^u). A flow is Anosov if these bundles can be chosen with $E^c = \langle \dot{f} \rangle$.

Some, but not necessarily all, distributions made up of these subbundles are integrable. For a point $x \in X$, the strong stable and strong unstable manifolds are defined by

$$W^{su}(x) = \{y \in X \mid d(f^{-t}x, f^{-t}y) \to 0 \text{ as } t \to +\infty\}$$
$$W^{ss}(x) = \{y \in X \mid d(f^t x, f^t y) \to 0 \text{ as } t \to +\infty\}.$$

These are C^∞-immersed submanifolds of X, with $T_x W^{su}(x) = E^u(x)$ and $T_x W^{ss}(x) = E^s(x)$. The strong unstable (resp. stable) leaves form a foliation of X, which we denote W^{su} (resp. W^{ss}).

If the flow is Anosov, the center bundle consists only of the flow direction and we can define weak-unstable (resp. weak-stable) manifolds through x by

$$W^u(x) = \bigcup_{t \in \mathbb{R}} W^{su}(f^t x)$$

$$W^s(x) = \bigcup_{t \in \mathbb{R}} W^{ss}(f^t x)$$

Then $T_x W^u(x) = E^u(x) \oplus E^c(x)$ and $T_x W^s(x) = E^s(x) \oplus E^c(x)$. In contrast, the center bundle of a partially hyperbolic flow may be non-integrable, and the existence of weak-stable and weak-unstable manifolds is not guaranteed.

In our setting, the geodesic flow on a manifold of negative sectional curvature is an Anosov flow, while the frame flow is an example of a partially hyperbolic flow with center bundle of dimension $1 + \dim SO(n-1)$. The frame flow actually has a stronger property that implies partial hyperbolicity: it acts *isometrically* on the center bundle, with respect to any bi-invariant metric on $SO(n-1)$.

2.3. **Pressure.** Given a function $\varphi : X \to \mathbb{R}$ (often called a *potential*), consider the accumulation of φ along orbits of f^t given by $\varphi_T(x) = \int_0^T \varphi(f^t(x))\, dt$. Let $B(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$, and let

$$B(x,\epsilon,T) = \{y \in X \ : \ d(f^t x, f^t y) < \epsilon \text{ for } 0 \leq t \leq T\}.$$

If $\bigcup_{x\in E} B(x,\epsilon,T) = X$ for some set $E \subset X$, then E is called (T,ϵ)-spanning. Then the value

$$S(f,\varphi,\epsilon,T) = \inf\{\sum_{x\in E} e^{\varphi_T(x)} \ : \ \bigcup_{x\in E} B(x,\epsilon,T) = X\}$$

gives the minimum accumulation of e^{φ} for time T of a (T,ϵ)-spanning set. The *topological pressure* of (f^t,φ) is the exponential growth rate (as $T \to \infty$) of this quantity as the resolution ϵ becomes finer:

$$P(f^t,\varphi) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log S(f,\varphi,\epsilon,T).$$

Note that when $\varphi \equiv 0$, the sum $\sum e^{\varphi_T x}$ simply counts the elements of the (T,ϵ) spanning set, and we recover $h_{top}(f^t)$. The *measure theoretic pressure*, with respect to an invariant measure μ, is defined to be

$$P_\mu(f^t,\varphi) = h_\mu(f^t) + \int_X \varphi\, d\mu.$$

For a Hölder continuous potential φ, the variational principle states that

$$P(f^t,\varphi) = \sup_{\mu\in\mathcal{M}(f^t)} P_\mu(f^t,\varphi).$$

A measure $\mu \in \mathcal{M}(f^t)$ for which $P_\mu(f^t,\varphi) = P(f^t,\varphi)$ is called an *equilibrium measure for* (f^t,φ).

2.4. **Conditional measures.** Given a probability space (Z,μ), a measurable partition determines a way to disintegrate the measure μ. Let $\mathcal{P}$ be a partition of Z into measurable sets. Let $\pi : Z \to \mathcal{P}$ be the map sending $z \in Z$ to the element $Q \in \mathcal{P}$ that contains it, and set $\hat{\mu} := \pi_*\mu$ (note that this is a measure on the partition $\mathcal{P}$). Then a *system of conditional measures* (relative to $\mathcal{P}$) is a family $\{\mu_Q\}_{Q\in\mathcal{P}}$ such that

(1) for each $Q \in \mathcal{P}$, the measure μ_Q is a probability measure on Q, and
(2) for each μ-measurable set $B \subset Z$, the map $Q \mapsto \mu_Q(B\cap Q)$ is $\hat{\mu}$-measurable and
$$\mu(B) = \int_{\mathcal{P}} \mu_Q(B\cap Q) d\hat{\mu}(Q).$$

We will often abbreviate this second statement by writing $\mu = \int_{\mathcal{P}} \mu_Q d\hat{\mu}$. By a theorem of Rokhlin, whenever $\mathcal{P}$ is a measurable partition, there exists a system of conditional measures relative to $\mathcal{P}$. It is a straight-forward consequence of item (2) that any two such systems must agree on a set of full $\hat{\mu}$-measure.

3. Equilibrium measures for fiber bundles

Let $\pi : Y \to X$ be a fiber bundle with Y a measurable metric space and fibers a compact Lie group V. Let $F^t : Y \to Y$ be a smooth flow, and let $\mathcal{M}(F^t)$ denote the set of F^t-invariant probability measures on Y. If F^t takes fibers to fibers and commutes with the action of the structure group (i.e., F^t is a bundle automorphism for all $t \in \mathbb{R}$), then there is a flow $f^t : X \to X$ such that $\pi \circ F^t = f^t \circ \pi$. In this case, an F^t-invariant probability measure $\mu \in \mathcal{M}(F^t)$ can be pushed forward to get an f^t-invariant probability measure $\hat{\mu} = \pi_*\mu \in \mathcal{M}(f^t)$ on X. Note that as long as the fibers are measurable sets, the partition $\{\pi^{-1}(x)\}_{x\in X}$ of Y is measurable.

In order to leverage information about equilibrium measures for the base dynamics, we use the assumption that the potential function on Y is constant on the fibers. The following illustrates the difficulties inherent in the more general case, and how this assumption resolves them. Let $\varphi : Y \to \mathbb{R}$ be a Hölder continuous function. Then, given a measure $\mu \in \mathcal{M}(F^t)$ and a disintegration of that measure $\mu = \int_X \mu_x d\hat{\mu}$, we can define a function $\hat{\varphi}_\mu : X \to \mathbb{R}$ by taking the average of φ on the fibers $\pi^{-1}(x)$:

$$\hat{\varphi}_\mu(x) = \int_{\pi^{-1}x} \varphi \, d\mu_x.$$

Since any two such disintegrations of μ agree on a full $\hat{\mu}$ measure set, any two functions defined by different disintegrations of μ agree on a set of full $\hat{\mu}$ measure. In general, the function $\hat{\varphi}_\mu : X \to \mathbb{R}$ is only measurable, since the disintegration $x \mapsto \mu_x$ is only measurable. In the case that φ is constant on fibers, however, $\hat{\varphi}_\mu$ is just the common value of φ and thus does not depend on the conditional measures of μ. Then the Hölder continuity of φ implies that $\hat{\varphi} = \hat{\varphi}_\mu$ is also Hölder continuous.

While the existence of equilibrium measures is in general a non-trivial problem, work of Newhouse and Yomdin shows that such a measure always exists for C^∞ dynamics. Since our Riemannian metric is C^∞, both the geodesic flow and frame flow are also C^∞, and so an equilibrium measure is guaranteed to exist [26]. Alternatively, one can use that the frame flow is entropy expansive and standard arguments to show the existence of an equilibrium measure (Theorem 2.4 of [15]). Thus, we are concerned in the following only with the uniqueness of equilibrium measures.

Since V is assumed to be a compact Lie group, it has a bi-invariant metric. We will say that $F : Y \to Y$ acts isometrically on the fibers if F preserves distances in the fibers with respect to such a metric.

Lemma 3.1. *Suppose that $F^t : Y \to Y$ acts isometrically on the fibers of the bundle $Y \to X$, and let φ be a Hölder continuous function that is constant on fibers. Let μ be an equilibrium measure for (F^t, φ). Then $\hat{\varphi}$ is a Hölder continuous function and $\hat{\mu} = \pi_*\mu$ is an equilibrium measure for $(f^t, \hat{\varphi})$.*

Proof. Since F^t acts isometrically on the fibers, $F^t|_{\pi^{-1}(x)}$ does not generate any entropy. Then the Ledrappier-Walters formula [21] implies that $h_\mu(F^t) = h_{\hat{\mu}}(f^t)$, and

$$\begin{aligned} P(F^t, \varphi) &= h_\mu(F^t) + \int_Y \varphi \, d\mu \\ &= h_{\hat{\mu}}(f^t) + \int_X \int_{\pi^{-1}x} \varphi \, d\mu_x d\hat{\mu}(x) \\ &\le P(f^t, \hat{\varphi}). \end{aligned}$$

Now suppose that ν is an equilibrium measure on X for $(f^t, \hat{\varphi})$, and let λ_x be (normalized) Haar measure on the fiber $\pi^{-1}(x)$. Then $x \mapsto \lambda_x$ is ν-measurable, and $\tilde{\nu} = \int_X \lambda_x d\nu(x) \in \mathcal{M}(F^t)$. Moreover, $\hat{\varphi}(x) = \int \varphi \, d\lambda_x$ because φ is constant on fibers. Thus,

$$\begin{aligned} P(f^t, \hat{\varphi}) &= h_\nu(f^t) + \int_X \Big(\int_{\pi^{-1}x} \varphi \, d\lambda_x\Big) d\nu(x) \\ &= h_{\tilde{\nu}}(F^t) + \int_Y \varphi \, d\tilde{\nu} \\ &\leq P(F^t, \varphi), \end{aligned}$$

so $P(f^t, \hat{\varphi}) = P(F^t, \varphi)$. Hence,

$$\begin{aligned} P_{\hat{\mu}}(f^t, \hat{\varphi}) &= h_{\hat{\mu}}(f^t) + \int_X \hat{\varphi} d\hat{\mu} \\ &= h_\mu(F^t) + \int_Y \varphi \, d\mu \\ &= P(F^t, \varphi) = P(f^t, \hat{\varphi}), \end{aligned}$$

so $\hat{\mu}$ is an equilibrium measure for $(f^t, \hat{\varphi})$. □

In order to study the conditional measures μ_x, we define subgroups of $\operatorname{Isom} V_x$ by their interaction with the measure μ_x. Let μ be a measure on Y with decomposition $\mu = \int \mu_x d\hat{\mu}$ along fibers, and let

$$G^\mu_x = \{\phi \in \operatorname{Isom} V_x \mid \phi_* \mu_x = \mu_x \text{ and } \phi(\xi g) = \phi(\xi) g \text{ for all } g \in V, \mu_x\text{-a.e. } \xi \in V_x\}.$$

This is clearly a subgroup of $\operatorname{Isom} V_x$. Moreover, since any two decompositions of μ agree on a set of full $\hat{\mu}$-measure, any two sets $\{G^\mu_x \mid x \in X\}$ defined using different decompositions of μ agree $\hat{\mu}$-almost everywhere.

The next lemma characterizes the support of conditional measures. It is an adaptation of Lemma 5.4 from [18].

Lemma 3.2. *Let $F : Y \to Y$ be a fiber bundle automorphism that acts by isometries on the fibers and suppose μ is an ergodic measure. Then for $\hat{\mu}$-almost every $x \in X$ and μ_x-almost every $\xi \in V_x$,*

$$G^\mu_x \xi = \mathit{supp}\, \mu_x.$$

Proof. First, we show that $G^\mu_x \xi \subseteq \operatorname{supp} \mu_x$. Fix an $x \in X$, let $\phi \in G^\mu_x$ and let $\xi \in \operatorname{supp} \mu_x$. Let $B_\epsilon \subset V_x$ be the ball of radius ϵ around $\phi\xi$. Then, by the definition of G^μ_x, we have $\phi_* \mu_x = \mu_x$ and

$$\mu_x(B_\epsilon) = \phi_* \mu_x(B_\epsilon) = \mu_x(\phi^{-1} B_\epsilon) > 0$$

because $\xi \in \phi^{-1} B_\epsilon$ and $\xi \in \operatorname{supp} \mu_x$. Hence, $\phi\xi \in \operatorname{supp} \mu_x$.

Next, we show that there is a set of full $\hat{\mu}$ measure in X such that for μ_x-almost every ξ, we have $G^\mu_x \xi \supseteq \operatorname{supp} \mu_x$. Let $\eta \in \operatorname{supp} \mu_x$; we will show that for μ_x-almost every ξ there is a $\phi \in G^\mu_x$ such that $\phi(\xi) = \eta$. Recall that the μ-disintegration $x \mapsto \mu_x$ is only a measurable map. By Lusin's Theorem, however, for any $\epsilon > 0$ there exists a closed set $K_\epsilon \subset X$ such that

(1) the map $K_\epsilon \to \mathcal{M}(E)$ taking x to μ_x is continuous, and
(2) $\hat{\mu}(K_\epsilon) > 1 - \epsilon$.

Since μ is an ergodic measure, the Birkhoff Ergodic Theorem implies that μ-almost every point in Y has dense orbit in $\operatorname{supp} \mu$. Thus, the set

$$K'_\epsilon := \{\xi \in K_\epsilon \,|\, \mu_x\text{-a.e.}\, \xi \in \pi^{-1}(x) \text{ has dense orbit in } \operatorname{supp} \mu\}$$

has measure $\hat{\mu}(K'_\epsilon) = \hat{\mu}(K_\epsilon) > 1 - \epsilon$. Moreover, for any $x \in K'_\epsilon$ and μ_x-almost any $\xi \in \pi^{-1}(x)$, there exists a sequence of times t_i such that $F^{t_i}\xi \to \eta$. Then:

(1) Since $F^{t_i}|_{\pi^{-1}(x)}$ are all isometries, $F^{t_i}|_{\pi^{-1}(x)} \to \phi \in \operatorname{Isom} V_x$.
(2) $\phi(\xi) = \eta$.
(3) By the F^t-invariance of μ, we have that $F^t_*\mu_x = \mu_{f^t x}$ for $\hat{\mu}$-almost every x. This, together with the continuity of the map $x \mapsto \mu_x$ on K'_ϵ, implies that

$$\phi_*\mu_x = \lim_{i\to\infty} F^{t_i}_*\mu_x = \lim_{i\to\infty} \mu_{f^{t_i}x} = \mu_x.$$

Further, since F^t commutes with the action of the isometry group on the fibers and $F^{t_i}|_{\pi^{-1}(x)} \to \phi$, we get that $\phi(\xi g) = \phi(\xi)g$. Hence, $\phi \in G^\mu_x$.

Letting $\epsilon \to 0$ gives a set K of full $\hat{\mu}$-measure such that for any $x \in K$, μ_x-almost every ξ satisfies $G^\mu_x\xi \supseteq \operatorname{supp}\mu_x$. $\square$

3.1. **Regularity of sections.** Let M be a closed manifold, and consider an Anosov flow $f^t : M \to M$. Then M has the following local product structure with respect to the strong stable foliation W^{ss} and weak unstable foliation W^u: for any $x \in M$, there exists a neighborhood V_x of x such that every point in V_x can be written as $[y,z] := W^u_{loc}(y) \cap W^{ss}_{loc}(z)$ for some $y \in W^{ss}_{loc}(x)$ and $z \in W^u_{loc}(x)$.

Let $\pi^\sigma_x : V_x \to W^\sigma_{loc}(x)$ (for $\sigma = ss, u$) be the projection map onto the appropriate local manifold. Given a measure μ on M, let $\mu^\sigma_x = (\pi^\sigma_x)_*\mu$ (for $\sigma = ss, u$). We say that μ *has local product structure* if

$$d\mu([y,z]) = \phi_x(y,z)\, d\mu^{ss}_x(y) \times d\mu^u_x(z)$$

for any $y \in W^{ss}_{loc}(x)$ and any $z \in W^u_{loc}(x)$, where ϕ_x is a non-negative Borel function.

Remark 3.3. The definition above corresponds to the one used in [22]. An alternate definition states that μ has local product structure if μ is *locally equivalent* to $\mu^{ss}_x \times \mu^u_x$ [3]. This alternate definition is a strictly stronger property, since the function ϕ_x above is only a *non-negative* Borel function.

Methods of [16] extend to prove the following Livšic regularity theorem.

Theorem 3.4. *Let $P \to M$ be a principal H-bundle over a compact, connected manifold M with H a compact group. Suppose $G = \mathbb{R}$ or $\mathbb{Z}$ acts Hölder continuously by bundle automorphisms such that the induced action on M is Anosov. Let m be a G-invariant measure on M with local product structure, and let $L \subseteq M$. Let V be a transitive left H-space that admits an H-invariant metric, and consider the associated bundle $E_V \to M$. Then any G-invariant measurable (w.r.t. m) section $L \to E_V$ is Hölder continuous on a subset $L' \subseteq L$ with $m(L') = m(L)$.*

Remark 3.5. Goetze and Spatzier in [16] prove this result for Anosov actions for an *invariant smooth measure* for Lie groups with a bi-invariant metric; the theorem above extends this to invariant measures with local product structure.

Remark 3.6. In particular, H is a transitive left H-space with H-invariant metric. In this case, the conclusion of the theorem states that any G-invariant measurable section $L \to P$ is Hölder continuous on a full measure subset of L.

The remainder of this section provides an outline of the ideas used to prove this theorem in the case $V = H$; the interested reader can find details in [16]. First, we discuss the relationship between sections of a bundle and cocycles. A *cocycle* is a map $\alpha : G \times M \to H$ such that

$$\alpha(t_2 + t_1, x) = \alpha(t_2, g_{t_1} x)\alpha(t_1, x).$$

Two such cocycles α and β are *cohomologous* if there is a function $\psi : M \to H$ such that

$$\alpha(t, x) = \psi(g_t x)\beta(t, x)\psi(x)^{-1}.$$

We say that α and β are *measurably* (resp. *Hölder*) *cohomologous* if ψ can be chosen to be a measurable (resp. Hölder continuous) function. If the underlying action is ergodic, then it follows that ψ is unique up to a constant. In this case, the regularity of the cohomology does not depend on the ψ chosen.

Measurable sections correspond under the G-action to measurable cocycles as follows. Let $\sigma : M \to P$ be a measurable section. This determines a measurable cocycle $\alpha : G \times M \to H$ by the relationship

$$g_t\sigma(x) = \sigma(g_t x)\alpha(t, x),$$

(α is uniquely determined since $P \to M$ is a principle H-bundle). Given a function $b : M \to H$, the section $\sigma_b(x) := \sigma(x)b(x)$ yields a cocycle $\beta(t, x)$ that is cohomologous to α. As σ is a measurable G-invariant section, α is in fact a measurable coboundary (i.e., measurably cohomologous to the trivial cocycle).

Since $P \to M$ may be a non-trivial bundle, there may be topological obstructions to a continuous section $M \to P$. Thus, in order to consider increasing the regularity of the section σ, we must break up the correspondence with cocycles into local trivializations on open sets U_i that cover M. On an open set U_i, there is a smooth section $s_i : U_i \to P$. Then, for any $x \in U_i$, there is a map $h_i : P \to H$ such that for any $x^* \in P$ in the fiber over $x \in U_i$,

$$x^* = s_i(x)h_i(x^*).$$

In particular, $\sigma(x) = s_i(x)h_i(\sigma(x))$. Note that the maps h_i are uniformly Lipschitz, but $h_i \circ \sigma$ is *a priori* only measurable since σ is only measurable.

The proof of the theorem thus reduces to showing that $h_{i(x)} \circ \sigma : L \to H$ is Hölder continuous. (This then implies that the section σ must also be Hölder continuous, since $h_{i(x)}$ is uniformly Lipschitz and its image is transverse to the fibers.) Let $L' \subseteq L$ be the set of points that are also in the support of m and for which the Birkhoff Ergodic Theorem holds. We have $m(L') = m(L)$. Consider two points $x, w \in L'$ that are close enough to be in a neighborhood of local product structure. Then, there exist $y \in W^{ss}_{loc}(x)$ and $z \in W^u_{loc}(x)$ such that $w = [y, z]_x$. Observe that also $x = [z, y]_w$.

By local product structure of m, we have

$$dm([y, z]) = \phi_x(y, z)\, dm^{ss}_x(y) \times dm^u_x(z)$$

for a non-negative Borel function ϕ_x. Since $w = [y, z]$ is in the support of m, ϕ must be positive at (y, z), y must be in the support of m^{ss}_x, and z must be in the support of m^u_x. Likewise, z must be in the support of m^{ss}_w and y must be in the support of m^u_w. Then, by the triangle inequality,

$$d(h_i(\sigma(x)), h_i(\sigma(w)) \leq d(h_i(\sigma(x)), h_i(\sigma(y))) + d(h_i(\sigma(y)), h_i(\sigma(w)))$$

(where d denotes distance in H). Thus, the problem is further reduced to showing that $h_{i(x)} \circ \sigma$ is Hölder continuous along stable and unstable manifolds.

Toward this end, consider two points $x, y \in L'$ on the same stable manifold. We want to measure the distance between $h_{i(x)}(\sigma(x))$ and $h_{i(y)}(\sigma(y))$. Recall that σ is a G-invariant section, and note that x and y are on the same stable manifold and so eventually are in the same open set U_j. Then the Hölder continuity of the G-action, along with the exponential contraction along the stable leaf, allow us to reduce our consideration to the distance between $h_j(g_t\sigma(x))$ and $h_j(g_t\sigma(y))$.

Although σ is a measurable section, Lusin's Theorem guarantees a compact set $K \subset L$ with $m(K) > 1/2$ on which σ is uniformly continuous. This implies that, for x and y in a set of full measure, there is an unbounded set of t such that $g_t\sigma(x)$ and $g_t\sigma(y)$ are in K. For such x and y, combining this with the previous paragraph gives Hölder continuity of $h_{i(x)} \circ \sigma$ along the local stable manifold. A similar argument shows that $h_{i(x)} \circ \sigma$ is Hölder continuous along the unstable manifold.

4. Proofs of theorems

Consider the frame flow $F^t : FM \to FM$ of a closed, oriented, negatively curved manifold M, and a Hölder continuous potential $\varphi : FM \to \mathbb{R}$ that is constant on the fibers of the bundle $FM \to SM$. As discussed above, there exists an equilibrium measure μ for (F^t, φ); we will show this equilibrium measure is unique.

Suppose μ is an equilibrium measure for (F^t, φ). By Lemma 3.1, $\hat{\mu}$ is an equilibrium measure for $(f^t, \hat{\varphi})$, and $\hat{\varphi}$ is Hölder continuous. The following result about equilibrium measures for hyperbolic flows then applies to $\hat{\mu}$:

Theorem 4.1 (Bowen-Ruelle, Leplaideur, [5], [22]). *Let $f^t : M \to M$ be an Anosov flow and $\varphi : M \to \mathbb{R}$ a Hölder continuous potential. Then there exists a unique equilibrium measure for (f^t, φ). It is ergodic and has local product structure and full support.*

4.1. Proof of Theorem 1 in dimension 3.

We first prove Theorem 1 in the case $n = 3$, where the logic of the proof is the same but the groups are simpler. The reason for this is that in the 3-dimensional case, the fibers of the bundle $FM \to SM$ are $\mathbb{S}^1$, which is an abelian group.

Consider the conditional measures $\{\mu_x\}$ given by Rokhlin decomposition $\mu = \int \mu_x d\hat{\mu}$. By Lemma 3.2, the support of a conditional measure μ_x for a typical point x is the orbit of a closed subgroup of isometries of the fiber. Since the fibers of $FM \to SM$ are $SO(2) \approx \mathbb{S}^1$, we get two possibilities: either

(1) $\operatorname{supp} \mu_x = \mathbb{S}^1$ for $\hat{\mu}$-almost every x, or
(2) μ_x is atomic and supported on m points for $\hat{\mu}$-almost every x.

In the first case, Lemma 3.2 implies that $G_x^\mu = \mathbb{S}^1$, so μ_x must be a multiple of Haar measure. Since $\hat{\mu}$ is ergodic, this multiple must be constant, and since μ is a probability measure, the constant must be one. Hence, $\mu = \int \mu_x \, d\hat{\mu}$ is uniquely determined, since $\hat{\mu}$ is unique by Theorem 4.1. We will show that topological considerations prevent the second case from occurring.

The easiest case is if $m = 1$. Let L be the full $\hat{\mu}$-measure set on which $\operatorname{supp} \mu_x$ is one point. This gives a measurable section $\sigma : L \longrightarrow FM$ sending x to the point $\operatorname{supp} \mu_x$. Since $\hat{\mu}$ has local product structure by Theorem 4.1, we can apply Theorem 3.4 to show that σ is actually Hölder continuous. Because $\hat{\mu}(L) = 1$ and $\hat{\mu}$ has full support on SM, σ can then be extended to a Hölder continuous section $SM \to FM$. Restricting this section to S_pM for some $p \in M$ gives a continuous map on $S_pM \approx \mathbb{S}^2$ that sends each point to an element of $SO(2) \approx \mathbb{S}^1$. This can be seen as a non-vanishing, continuous vector field on $\mathbb{S}^2$, which is a contradiction.

Now suppose that $m > 1$. Let F be the fiber of the bundle $FM \to SM$. Construct a new bundle $FM^m \to SM$ with fibers the Cartesian product F^m. Then the discussion above produces a measurable map $\sigma : L \to FM^m/\Sigma_m$, where Σ_m acts on FM^m by permutations, sending $x \in L$ to the m points in the support of μ_x. Now we apply Theorem 3.4 to the associated bundle $E_V = FM^m/\Sigma_m$ with $H = SO(2)^m$ and $V = SO(2)^m/\Sigma_m$. This implies that σ is a Hölder continuous map on L, which then can be extended to a Hölder continuous map $M \to FM^m/\Sigma_m$. Then the projection map $FM^m/\Sigma_m \to SM$, restricted to the preimage of a set S_pM is an m-fold cover of $S_pM \approx \mathbb{S}^2$. Since $\mathbb{S}^2$ is simply connected, the preimage must be a disjoint union of m copies of $\mathbb{S}^2$. Restricting the cover to one of these copies gives a non-vanishing, continuous vector field on $\mathbb{S}^2$ as in the case $m = 1$, which is a contradiction.

4.2. **Proof of Theorem 1 in higher dimensions.** More generally, the structure group of $FM \to SM$ is $SO(n-1)$. Recall that the measure μ_x is invariant under the group G_x^μ by construction. Then either $G_x^\mu = SO(n-1)$ for $\hat{\mu}$-almost every x and μ is uniquely determined by $\hat{\mu}$ (as in Section 4.1), or else G_x^μ is a strict subgroup of $SO(n-1)$ $\hat{\mu}$-almost everywhere.

Again, let us show that the second case cannot occur. By ergodicity of $\hat{\mu}$, G_x^μ must be the same subgroup $H < SO(n-1)$ $\hat{\mu}$-almost everywhere. This gives a measurable section $\sigma : SM \to FM/H$ that takes x to the support of μ_x on a set of full $\hat{\mu}$ measure. By Theorem 3.4, we can extend this to a continuous, global section $SM \to FM/H$. Such a section gives a reduction of the structure group of $FM \to SM$, as follows. The section σ provides a trivialization of the bundle $FM/H \to SM$. Since $FM \to FM/H$ is a principle H-bundle, the the pullback $\sigma^*(FM) = FM_H$ is a reduction of FM with structure group H.

$$\begin{tikzcd} FM_H \arrow[r, dotted] \arrow[d, dotted] & FM \arrow[d, "\pi"] \\ SM \arrow[r, "\sigma"] & FM/H \end{tikzcd}$$

A non-trivial reduction of the structure group of $FM \to SM$ also gives a non-trivial reduction of the structure group of the restricted bundle $F_pM \to S_pM \approx \mathbb{S}^{n-1}$ over a point $p \in M$. Then, for n odd and not equal to 7, Proposition 5.1 of [6] implies that H cannot act transitively on $\mathbb{S}^{n-2} = \pi_2^{-1}(p)$ (the fiber of the projection of 2-frames to the unit tangent bundle over a point p). However, for n odd, any such structure group must act transitively on $\mathbb{S}^{n-2}$, by Corollary 4.2 of [6]. This is a contradiction. Thus, G_x^μ must be equal to $SO(n-1)$ μ_L-a.e., and μ is uniquely determined. □

4.3. **Proof of Theorem 2.** For the Heisenberg manifold M, the bundle $M \to \mathbb{T}^2$ is a fiber bundle with structure group $\mathbb{S}^1$. Any automorphism of M automatically preserves the $\mathbb{S}^1$ fibers, acts by volume preserving diffeomorphisms on both M and $\mathbb{T}^2$, and hence has eigenvalue 1 in the (invariant) fiber direction. Thus f acts by isometries on the fibers. By assumption, f induces an Anosov diffeomorphism on the base torus $\mathbb{T}^2$. Let $\varphi : M \to \mathbb{R}$ be a Hölder continuous potential function that is constant on the fibers of $M \to \mathbb{T}^2$, and let m be an equilibrium measure for

(f, φ). Then the arguments from the case of frame flows apply verbatim and give us the dichotomy: either m is invariant under the structure group $\mathbb{S}^1$, and we can understand the equilibrium state via the base torus $\mathbb{T}^2$, or there is a continuous invariant section of the fibre bundle $M \to \mathbb{T}^2$ or a finite cover. In the first case, m is uniquely determined since the equilibrium measure of an Anosov diffeomorphism on $\mathbb{T}^2$ is unique (Theorem 4.1 of [4]). The second case implies that for a finite cover $\bar{M}$ of M, $\pi_1(\bar{M}) = \mathbb{Z} \times \mathbb{Z}^2$, which is impossible since $Heis(\mathbb{Z})$ does not contain a rank 3 abelian subgroup.

4.4. **Proof of Theorem 3.** The following proof of Theorem 3 replaces an earlier, longer argument and was suggested by the referee.

Consider the so-called *geometric potential* defined by

$$\phi^u = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{Jac}_u F^t,$$

where Jac_u is the Jacobian of the restriction of the geodesic flow to the unstable manifold. Since the frame flow acts isometrically on fibers, ϕ^u is constant along the fibers. Moreover, since the metric g is smooth and the unstable foliation is Hölder, ϕ^u is Hölder continuous. Hence, by Theorem 1, the equilibrium measure is unique. Following the arguments in Section 4.2, the conditional measures must be invariant by the structure group $SO(n-1)$, and μ projects to the unique equilibrium measure for the geometric potential for the geodesic flow. This is well-known to be the Liouville measure (Exercise 20.4.1, [17]). Hence, μ is the natural smooth measure for the frame flow, and it is ergodic by Theorem 1. □

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