TOTALLY NON-SYMPLECTIC ANOSOV ACTIONS ON TORI AND NILMANIFOLDS

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ABSTRACT. We show that sufficiently irreducible totally non-symplectic Anosov actions of higher rank abelian groups on tori and nilmanifolds are C^∞-conjugate to affine actions.

1. INTRODUCTION

Hyperbolic actions of abelian groups of rank at least 2 exhibit many surprising rigidity properties. Case in point is the local smooth rigidity of actions by automorphisms of tori and nilmanifolds and other algebraically defined actions. This means that perturbations of an action that are C^1-close for a finite set of generators are C^∞-conjugate to the original action. It was established for algebraic actions with semisimple linear part by Katok and Spatzier in [18] and for some non-semisimple action on tori by Einsiedler and T. Fisher [3]. The higher rank situation is entirely different from the case of single Anosov diffeomorphisms and flows for which it is always easy to construct C^1-small perturbations which are not even C^1-conjugate.

Local smooth rigidity of algebraic actions gives strong support to the following conjecture by Katok and Spatzier.

Classification Conjecture: All "irreducible" Anosov $\mathbb{Z}^k$ and $\mathbb{R}^k$-actions for $k \geq 2$ on any compact manifold are C^∞-conjugate to algebraic actions.

Kalinin and Spatzier proved this conjecture for the special class of Cartan actions of abelian groups of rank at least 3 under some other more technical hypotheses [15]. Here we call an action *Cartan* if maximal intersections of stable manifolds of various elements, called *coarse Lyapunov foliations*, are one-dimensional and, together with the orbit, span the space. Kalinin and Sadovskaya have results for more general Anosov actions of rank at least 2 where the condition on dimension 1 is replaced by either uniform quasi-conformality or a pinching condition [13, 14]. The basic idea of the proofs in all of these results is to build smooth structures on various foliations and then combine them. Unfortunately, this only works under strong assumptions on the action.

The general case of the conjecture remains out of reach. Thus it is natural to restrict attention to actions on tori and nilmanifolds where one usually refers to the conjecture as *global rigidity*. For these spaces, the classical results of Franks and Manning [5, 20] offer a different approach. Their work implies that any action α of an abelian group with at least one Anosov element on a torus or a nilmanifold is always C^0-conjugate to an action by affine Anosov automorphisms by some Hölder conjugacy ϕ. We call the latter action the *linearization* of α and refer to Section 2 for a precise definition. On the torus the linearization is essentially given by the induced action on homology. Now to prove global rigidity it suffices to show smoothness of the conjugacy ϕ.

The idea that a C^0 conjugacy can be used to get C^∞-rigidity appears already in Hurder's work on deformation rigidity of lattice actions on tori [9] and later in Katok-Lewis [17] for both their local and global rigidity theorems for Cartan actions on tori. It also formed the basis of the argument for local rigidity in Katok-Spatzier [18]. In the different context of local rigidity of algebraic actions of lattices in higher rank groups, work of Katok and Spatzier and later Fisher, Margulis and Qian [4, 18, 21] also involves finding a C^0 conjugacy that is improved to C^∞ using the presence of higher rank abelian subgroups in the acting group. Rodriguez Hertz established global rigidity for $\mathbb{Z}^k$ actions on tori with at least one Anosov element whose linearization has coarse Lyapunov foliations of dimensions one or two and either has maximal rank or satisfies additional bunching assumptions [25]. To date however, all results require that the derivatives of either the action or its linearization along the coarse Lyapunov foliations satisfy a pinching assumption. This means that the ratio of maximal over minimal contraction is controlled, e.g. less than 2. In this paper, we overcome this problem for the first time by a combination of the use of non-stationary normal forms and holonomy arguments.

Continuous normal forms were already introduced for the proof of local rigidity in [18]. In essence they give coordinate charts in which the derivatives of the map along contracting foliations take values in a finite dimensional Lie group. Moreover, the dependence of the coordinates on the base point is continuous in the C^∞-topology. Existence of continuous normal forms is guaranteed if the derivatives of the maps under consideration satisfy a spectral gap condition along the given contracting foliation. While such spectral gaps are automatic for C^1-perturbations of algebraic systems and also for one dimensional foliations, they fail to hold in general. In particular we cannot assume such spectral gaps for the proof of global rigidity. Instead, we use a measurable version of the non-stationary normal forms theory where the "measurable" spectral gap condition is always satisfied by Oseledec' Multiplicative Ergodic Theorem.

Let us next summarize some elements from the structure theory of higher rank abelian actions, see Section 2 for more details. They preserve a probability measure of full support. One can find a common Lyapunov splitting of the tangent bundle

$TM = \oplus_I E_i$ which refines the Lyapunov splittings of each individual element. Moreover, if $v \in E_i$, the Lyapunov exponent of v defines a linear functional, the *Lyapunov functional*, on the acting $\mathbb{Z}^k$ which we think of as a linear functional on the ambient $\mathbb{R}^k$. For affine actions by automorphisms the Lyapunv exponents are nothing but the logarithms of the absolute values of the eigenvalues of the automorphisms. A *Weyl chamber* is a connected component of $\mathbb{R}^k$ minus all the hyperplane kernels of the Lyapunov functional. We will need to make an assumption that every Weyl chamber defined by the linearization contains an Anosov element in the non-linear action. As we will later see that the Weyl chambers on the two sides agree, we abbreviate this by saying that every Weyl chamber contains an Anosov element. This allows us to define the *coarse Lyapunov foliations* as the maximal intersections of stable foliations of Anosov elements. Hence these foliations are Hölder with smooth leaves.

Recall that a matrix is semisimple if it is diagonalizable over $\mathbb{C}$. We call an action by automorphisms on a nilmanifold *semisimple* if every element acts by a semisimple matrix.

Finally, we call a $\mathbb{Z}^k$-action *TNS* or *totally non-symplectic* if any two $v \in E_i$ and $w \in E_j$ belong to the stable distribution of some element $a \in \mathbb{Z}^k$. This excludes the possibility of a bilinear form invariant under the action, hence the name.

The main result of this paper is global rigidity for totally non-symplectic actions of higher rank abelian groups for which sufficiently many elements are Anosov.

Theorem 1.1. *Suppose α is a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$ on a nilmanifold N/Γ. Assume the linearization ρ_0 of α is semisimple and TNS and there is an Anosov element in each Weyl chamber of α. Then α is C^∞-conjugate to an affine action with linear part ρ_0.*

As discussed above, this theorem is the first that does not require pinching conditions. Moreover, it also yields the first global rigidity result for Anosov actions on nilmanifolds which are not tori. Indeed, in all earlier results the pinching condition, together with various additional assumptions such as integrability or absence of certain resonances, forced the nilmanifold to be a torus.

Call a linear $\mathbb{Z}^k$ action on a torus *totally reducible* if every rational invariant torus has a rational invariant complement. There is a similar though more complicated notion for nilmanifolds which we describe below in section 9. We will show that total reducibility is equivalent to semisimplicity, and thus we immediately get the next result:

Corollary 1.2. *Suppose α is a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$ on a nilmanifold N/Γ. Assume the linearization ρ_0 of α is totally reducible and TNS and there is an Anosov element in each Weyl chamber of α. Then α is C^∞-conjugate to an affine action with linear part ρ_0.*

Our results have some applications to global rigidity for actions of higher rank lattices. It is a theorem of Margulis and Qian that any Anosov action of a higher rank lattice Γ on a nilmanifold (with a global fixed point) is continuously conjugate to an affine action [21]. It is well known that the Γ contains many abelian subgroups isomorphic $\mathbb{Z}^k$, where k is the rank of Γ and that the Anosov Γ action restricts to an Anosov $\mathbb{Z}^k$ action for some k. If any $\mathbb{Z}^k$ subgroup satisfies the conditions of Theorem 1.1, it then follows from our results that the conjugacy is smooth, and therefore that the full Γ action is smoothly conjugate to an affine action.

Let us briefly indicate the main elements in the proof of Theorem 1.1. As discussed above we show that the topological conjugacy ϕ is smooth. For this, we first suspend the $\mathbb{Z}^k$-action to an $\mathbb{R}^k$-action. Then we fix a coarse Lyapunov foliation and for almost every leaf we construct a transitive group of smooth transformations which is intertwined by ϕ with the group of translations of the corresponding leaf for the linearization. As in other proofs of rigidity theorems e.g. in [18], we use limits of return maps. Unlike earlier proofs however, we do not directly use the acting group but rather holonomies along transversal coarse Lyapunov foliations. First we show that these holonomies are smooth. For this we establish existence of elements which contract the fixed coarse Lyapunov foliation slower than a transversal one. Then we show that the holonomies centralize suitable elements of $\mathbb{R}^k$ and hence preserve measurable non-stationary normal forms. It follows that limits of such holonomies are still smooth and define the desired transitive group actions. Once the smoothness of ϕ is established for a.e. leaf of each coarse Lyapunov foliation, the smoothness of holonomies gives the global smoothness of ϕ. A more detailed outline of the proof is given in Section 3, after all relevant notions have been defined.

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2. Preliminaries

Throughout the paper, the smoothness of diffeomorphisms, actions, and manifolds is assumed to be C^∞, even though all definitions and some of the results can be formulated in lower regularity.

2.1. Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$.

Let a be a diffeomorphism of a compact manifold M. We recall that a is *Anosov* if there exist a continuous a-invariant decomposition of the tangent bundle $TM = E_a^s \oplus E_a^u$ and constants $K > 0$, $\lambda > 0$ such that for all $n \in \mathbb{N}$

$$
\begin{aligned}
\|Da^n(v)\| &\leq Ke^{-\lambda n}\|v\| \ \text{ for all } \ v \in E_a^s, \\
\|Da^{-n}(v)\| &\leq Ke^{-\lambda n}\|v\| \ \text{ for all } \ v \in E_a^u.
\end{aligned}
\tag{1}
$$

The distributions E_a^s and E_a^u are called the *stable* and *unstable* distributions of a.

Now we consider a $\mathbb{Z}^k$ action α on a compact manifold M via diffeomorphisms. The action is called *Anosov* if there is an element which acts as an Anosov diffeomorphism. For an element a of the acting group we denote the corresponding diffeomorphisms by $\alpha(a)$ or simply by a if the action is fixed.

For a $\mathbb{Z}^k$ action α on a manifold M, there is an associated $\mathbb{R}^k$ action $\tilde{\alpha}$ on a manifold $\mathcal{S}$ given by the standard suspension construction [11]. Briefly, this is the action of $\mathbb{R}^k$ by left translations on $(\mathbb{R}^k \times M)/\mathbb{Z}^k$. Here $(\mathbb{R}^k \times M)/\mathbb{Z}^k$ is the quotient of $\mathbb{R}^k \times M$ by the $\mathbb{Z}^k$-action of $\mathbb{R}^k \times M$ given by $z(r, p) = (r - z, z(p))$. We will refer to $\tilde{\alpha}$ as the *suspension* of α. It generalizes the suspension flow of a diffeomorphism. Similarly, the manifold $\mathcal{S}$ is a fibration over the "time" torus $\mathbb{T}^k$ with the fiber M.

Definition 2.1. Let α be a smooth action of $\mathbb{R}^k$ on a compact manifold M. An element $a \in \mathbb{R}^k$ is called *Anosov* or *normally hyperbolic* for α if there exist positive constants λ, K and a continuous α-invariant splitting of the tangent bundle

$$TM = E_a^s \oplus E_a^u \oplus T\mathcal{O}$$

where $T\mathcal{O}$ is the tangent distribution of the $\mathbb{R}^k$-orbits, and (1) holds for all $n \in \mathbb{N}$.

An $\mathbb{R}^k$ action is called *Anosov* if some element $a \in \mathbb{R}^k$ is Anosov. Note that if $a \in \mathbb{Z}^k$ is Anosov for α if and only if it is Anosov for $\tilde{\alpha}$. Thus if α is an Anosov $\mathbb{Z}^k$ action then $\tilde{\alpha}$ is an Anosov $\mathbb{R}^k$ action.

Both in the discrete and the continuous case it is well-known that the distributions E_a^s and E_a^u are Hölder continuous and tangent to the stable and unstable foliations $\mathcal{W}_a^s$ and $\mathcal{W}_a^u$ respectively [8]. The leaves of these foliations are C^∞ injectively immersed Euclidean spaces. Locally, the immersions vary continuously in the C^∞ topology. In general, the distributions E^s and E^u are only Hölder continuous transversally to the corresponding foliations.

2.2. **Lyapunov exponents and coarse Lyapunov distributions.**

We will concentrate on the case of $\mathbb{R}^k$ actions, the case of $\mathbb{Z}^k$ is similar. We refer to [15] and [13] for more details.

Let a be a diffeomorphism of a compact manifold M preserving an ergodic probability measure μ. By Oseledec' Multiplicative Ergodic Theorem, there exist finitely many numbers χ_i and a measurable splitting of the tangent bundle $TM = \bigoplus E_i$ on a set of full measure such that the forward and backward Lyapunov exponents of $v \in E_i$ are χ_i. This splitting is called *Lyapunov decomposition*.

Let μ be an ergodic probability measure for an $\mathbb{R}^k$ action α on a compact manifold M. By commutativity, the Lyapunov decompositions for individual elements of $\mathbb{R}^k$ can be refined to a joint invariant splitting for the action. The following proposition from [15] describes the Multiplicative Ergodic Theorem for this case. See [13] for the discrete time version and [11] for more details on the Multiplicative Ergodic Theorem and related notions for higher rank abelian actions.

Proposition 2.2. *Let α be a smooth action of $\mathbb{R}^k$ and let μ be an ergodic invariant measure. There are finitely many linear functionals χ on $\mathbb{R}^k$, a set of full measure $\mathcal{P}$, and an α-invariant measurable splitting of the tangent bundle $TM = T\mathcal{O} \oplus \bigoplus E_\chi$ over $\mathcal{P}$, where $\mathcal{O}$ is the orbit foliation, such that for all $a \in \mathbb{R}^k$ and $v \in E_\chi$, the Lyapunov exponent of v is $\chi(a)$, i.e.*

$$\lim_{n \to \pm\infty} t^{-1} \log \|D\alpha(ta)v\| = \chi(a),$$

where $\|..\|$ is a continuous norm on TM.

The splitting $\bigoplus E_\chi$ is called the *Lyapunov decomposition*, and the linear functionals χ are called the *Lyapunov exponents* of α. The hyperplanes $\ker \chi \subset \mathbb{R}^k$ are called the *Lyapunov hyperplanes* or *Weyl chamber walls*, and the connected components of $\mathbb{R}^k - \cup_\chi \ker \chi$ are called the *Weyl chambers* of α. The elements in the union of the Lyapunov hyperplanes are called *singular*, and the elements in the union of the Weyl chambers are called *regular*. We note that the corresponding notions for a $\mathbb{Z}^k$ action and for its suspension are directly related. In particular, the nontrivial Lyapunov exponents are the same. In addition, for the suspension there is one identically zero Lyapunov exponent corresponding to the orbit distribution. From now on, the term Lyapunov exponent will always refer to the nonzero functionals.

Consider a $\mathbb{Z}^k$ action by *automorphisms* of a torus or an infranilmanifold $M = N/\Gamma$. In this case, the Lyapunov decomposition is determined by the eigenspaces of the automorphisms, and the Lyapunov exponents are the logarithms of the moduli of the eigenvalues. Hence they are independent of the invariant measure, and they give uniform estimates of expansion and contraction rates. Also, every Lyapunov distribution is smooth and integrable.

In the nonalgebraic case, the individual Lyapunov distributions are in general only measurable and depend on the given measure. This can be observed already for a single Anosov diffeomorphism. However, the full stable distribution for any measure always agrees with E_a^s. For higher rank actions, *coarse Lyapunov distributions* play a similar role. For any Lyapunov functional χ the coarse Lyapunov distribution is the direct sum of all Lyapunov spaces with Lyapunov functionals positively proportional to χ:

$$E^\chi = \oplus E_{\chi'}, \quad \chi' = c\chi \text{ with } c > 0.$$

One can see that for an algebraic action such a distribution is a finest nontrivial intersection of the stable distributions of certain Anosov elements of the action. For nonalgebraic actions, however, it is not a priori clear. It was shown in [15, Proposition 2.4] that, in the presence of sufficiently many Anosov elements, the coarse Lyapunov distributions are well-defined, continuous, and tangent to foliations with smooth leaves (see Proposition 2.2 in [14] for the discrete time case). We denote the set of all Anosov elements in $\mathbb{Z}^k$ or $\mathbb{R}^k$ by $\mathcal{A}$.

Proposition 2.3. *Let α be an Anosov action of $\mathbb{Z}^k$ or $\mathbb{R}^k$ and let μ be an ergodic probability measure for α with full support. Suppose that there exists an Anosov element in every Weyl chamber defined by μ. Then for each Lyapunov exponent χ the coarse Lyapunov distribution can be defined as*

$$E^\chi(p) = \bigcap_{\{a \in \mathcal{A} \,|\, \chi(a)<0\}} E_a^s(p) = \bigoplus_{\{\chi'=c\chi \,|\, c>0\}} E_{\chi'}(p)$$

on the set $\mathcal{P}$ of full measure where the Lyapunov exponents exist. Moreover, E^χ is Hölder continuous, and thus it can be extended to a Hölder distribution tangent to the foliation $\mathcal{W}^\chi = \bigcap_{\{a \in \mathcal{A} \,|\, \chi(a)<0\}} \mathcal{W}_a^s$ with uniformly C^∞ leaves.

Note that ergodic measures with full support always exist if a $\mathbb{Z}^k$ action contains a transitive Anosov element. A natural example is given by the measure μ of maximal entropy for such an element which, by uniqueness, is invariant under the action.

The action is called **totally nonsymplectic**, or **TNS**, if there are no negatively proportional Lyapunov exponents. For such an action, any pair of coarse Lyapunov distributions is contracted by some element.

2.3. $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions on tori and nilmanifolds.

Let f be an Anosov diffeomorphism of a torus or, more generally, a nilmanifold $M = N/\Gamma$. By the results of Franks and Manning in [5, 20], f is topologically conjugate to an Anosov automorphism $A : M \to M$, i.e. there exists a homeomorphism $\phi : M \to M$ such that $A \circ \phi = \phi \circ f$. The conjugacy ϕ is unique in the homotopy class of identity and is bi-Hölder, i.e. both ϕ and ϕ^{-1} are Hölder continuous with some Hölder exponent γ.

Now we consider an Anosov $\mathbb{Z}^k$ action α on a nilmanifold M. Fix any Anosov element a for α. Then we have ϕ which conjugates $\alpha(a)$ to an automorphism. It is well known that ϕ then conjugates α to an action ρ by affine automorphisms ([9], proof of Proposition 2.18). This follows from the fact that any homeomorphism commuting with an Anosov automorphism is an affine automorphism ([23], proof of Proposition 0). By an affine automorphism we mean a composition of an automorphism and a translation. We note that an Anosov $\mathbb{Z}^k$ action on an infranilmanifold may have no fixed points [9], however, there is always a finite index subgroup which fixes a point and whose action is conjugate to an action by automorphisms.

We denote by λ the normalized Haar measure on the infranilmanifold $\mathcal{N}$ and let $\mu = \phi_*^{-1}(\lambda)$. Then μ is invariant under α and for any Anosov element $a \in \mathbb{Z}^k$, μ is the measure of maximal entropy for $\alpha(a)$, since λ is for $\rho(a)$.

Remark. In fact, μ is absolutely continuous (cf. [12, Remark 1]). While we will not need this fact, here is a brief argument. The Jacobian of μ under α is a Hölder cocycle over ρ as Φ is Hölder. Thus they are cohomologous to a constant function due to Hölder cocycle rigidity of irreducible affine $\mathbb{Z}^k$-actions [18]. Hence μ is absolutely continuous by a well-known result on Jacobians of Anosov diffeomorphisms [16, Theorem 19.2.7]. By a similar argument the Jacobian along the unstable foliation

of an Anosov element is cohomologus to a constant function. By [16, Theorem 20.4.1] μ is the equilibrium state of the constant function. Since the measure of maximal entropy is the equilibrium state of the constant function [16, Theorem 20.1.3], and the equilibrium state is unique [16, Theorem 20.3.7] we conclude that they coincide.

We will show below that the Lyapunov exponents of (α, μ) and (ρ, λ) are positively proportional and that the corresponding coarse Lyapunov foliations are mapped into each other by the conjugacy ϕ.

Now we consider the suspensions $\tilde{\alpha}$ and $\tilde{\rho}$ of α and ρ. These are smooth $\mathbb{R}^k$ actions on the suspension manifolds $\mathcal{S}$ and $\mathcal{R}$ of α and ρ. We denote the lifts to the suspensions of the conjugacy and the invariant measures by $\tilde{\phi}$, $\tilde{\mu}$, and $\tilde{\lambda}$. Note that $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ are also Hölder continuous with the same exponent $\gamma > 0$ as ϕ and ϕ^{-1}.

From now on, instead of indexing a coarse Lyapunov by a representative of the class of positively proportional Lyapunov functionals, we index them numerically. I.e. we write $\mathcal{W}^i$ instead of $\mathcal{W}^\chi$, implicitly identifying the finite collection of equivalence classes of Lyapunov exponents with a finite set of integers. The next proposition summarizes important properties of the suspension actions. Similar properties hold for the original $\mathbb{Z}^k$ actions.

Proposition 2.4. *Assume there is an Anosov element in every Weyl chamber. Then*

(1) *The Lyapunov exponents of $(\tilde{\alpha}, \tilde{\mu})$ and $(\tilde{\rho}, \tilde{\lambda})$ are positively proportional, and thus the Lyapunov hyperplanes and Weyl chambers are the same.*

(2) *For any coarse Lyapunov foliation $\mathcal{W}^i_{\tilde{\alpha}}$ of $\tilde{\alpha}$*

$$\tilde{\phi}(\mathcal{W}^i_{\tilde{\alpha}}) = \mathcal{W}^i_{\tilde{\rho}},$$

where $\mathcal{W}^i_{\tilde{\alpha}}$ is the corresponding coarse Lyapunov foliation for $\tilde{\rho}$.

Remark. We note that the actual Lyapunov exponents of $(\tilde{\alpha}, \tilde{\mu})$ and $(\tilde{\rho}, \tilde{\lambda})$ (or of different invariant measures for $\tilde{\alpha}$) could be different.

Remark. In fact, one can show that the same holds for Lyapunov exponents and coarse Lyapunov foliations of (α, ν) for any α-invariant measure ν so, in particular, the Lyapunov exponents of all α-invariant measures are positively proportional and the coarse Lyapunov splittings are consistent with the continuous one defined in Proposition 2.3.

Proof: Note that the stable foliations of $\tilde{\alpha}(a)$ and $\tilde{\rho}(a)$ must be mapped into each other by the conjugacy $\tilde{\phi}$ since the stable manifolds are characterized by the contraction property. Hence the formula for $\mathcal{W}^i_{\tilde{\alpha}}$ given in Proposition 2.3 implies (2) once we establish (1).

To establish (1) we need to show that the (oriented) Lyapunov hyperplanes for $(\tilde{\alpha}, \tilde{\mu})$ and $(\tilde{\rho}, \tilde{\lambda})$ are the same. If a Lyapunov hyperplane L of one action is not a Lyapunov hyperplane of the other we get a contradiction: for close elements a, b across L their (local) stable manifolds for the second action are the same and

thus coincide with their intersection, for the first action the (local) stable manifold-sare different and their intersection has smaller dimension, but the intersections are homeomorphic by $\tilde{\phi}$. ⋄

3. Outline of the proof of Theorem 1.1

Proposition 2.3 shows that coarse Lyapunov foliations for α and $\tilde{\alpha}$ are well-defined continuous foliations with smooth leaves. By Proposition 2.4 they are mapped by the conjugacy to the corresponding homogeneous foliations for ρ and $\tilde{\rho}$. The main goal is to study the regularity of the conjugacy ϕ along these foliations.

For the most of the proof we consider a coarse Lyapunov foliation $\mathcal{W}$ for the suspension action $\tilde{\alpha}$. The first major step is to establish smoothness of certain holonomies between leaves of $\mathcal{W}$. The TNS assumption gives the existence of invariant foliations $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $T\mathcal{W}_1 \oplus T\mathcal{W} \oplus T\mathcal{W}_2 \oplus T\mathcal{O} = TM$. Moreover, each $T\mathcal{W}_i \oplus T\mathcal{W}$ is the stable distribution of some element and, in particular, is integrable. In Section 5 we show that the holonomies along $\mathcal{W}_1$ (and along $\mathcal{W}_2$) between leaves of $\mathcal{W}$ are C^∞. This follows from the existence of an element for which $\mathcal{W}_1$ a fast stable foliation inside $T\mathcal{W}_1 \oplus T\mathcal{W}$. To obtain such an element we establish in Section 4 that the expansion or contraction of $\mathcal{W}$ by an element in the corresponding Lyapunov hyperplane is uniformly slow.

The second major step is to establish smoothness of the conjugacy ϕ along the leaves of the coarse Lyapunov foliation $\mathcal{W}$. For this we introduce in Section 6 the measurable normal forms for the action on $\mathcal{W}$ defined a.e. with respect to the measure $\mu = \phi_*^{-1}(\lambda)$. In Section 7 we show that the smooth holonomies along of $\mathcal{W}_1$ preserve the normal forms on $\mathcal{W}$. For this we use the semisimplicity assumption to split the homogeneous foliation $\tilde{\phi}(\mathcal{W}_1)$ into subfoliations corresponding to eigenspaces for $\tilde{\rho}$. Then we see that holonomies along a particular subfoliation preserve the normal forms since they commute with an element in $\mathbb{R}^k$ which fixes the corresponding eigenspace and contracts $\mathcal{W}$. Since $\mathcal{W}_1$ is the full stable foliation of some element, it is ergodic with respect to μ, and hence the holonomies along a typical leaf are sufficiently transitive. Using this we show in Section 8 that for a typical leaf W of $\mathcal{W}$ and for almost every translation T of the homogeneous leaf $\phi(L)$, the conjugate map $\phi^{-1} \circ T \circ \phi : W \to W$ can be obtained as certain limit of such holonomies which also preserves the normal forms and therefore is smooth. This yields that ϕ is C^∞ along W.

Since the holonomies between different leaves of $\mathcal{W}$ along $\mathcal{W}_1$ and $\mathcal{W}_2$ are smooth and intertwine the restriction of ϕ to these leaves we obtain that ϕ is C^∞ along all leaves of $\mathcal{W}$ and that the derivatives are continuous transversally. Then the standard elliptic theory implies that ϕ is C^∞ on M.

4. Uniform estimates for elements near Lyapunov hyperplane

We consider the suspension actions $\tilde{\alpha}$ and $\tilde{\rho}$ of $\mathbb{R}^k$ on $\mathcal{S}$ and $\mathcal{R}$. We fix a Lyapunov hyperplane $L \subset \mathbb{R}^k$ and the corresponding positive Lyapunov half-space L^+. We denote the corresponding coarse Lyapunov distributions for $\tilde{\alpha}$ and $\tilde{\rho}$ by E and $\bar{E}$ respectively. Recall that $\gamma > 0$ denotes a Hölder exponent of $\tilde{\phi}$ and $\tilde{\phi}^{-1}$.

Lemma 4.1. *Consider an element $b \in \mathbb{R}^k$. Let $\bar{\chi}(b)$ be the largest Lyapunov exponent of $\tilde{\rho}(b)$ corresponding to $\bar{E}$ and denote $\chi_M = \max\{0, \bar{\chi}(b)/\gamma\}$. Let ν be any ergodic invariant measure for $\tilde{\alpha}(b)$ and let $\chi_\nu(b)$ be the largest Lyapunov exponent of $(\tilde{\alpha}(b), \nu)$ corresponding to the distribution E. Then $\chi_\nu(b) \leq \chi_M$*

Proof: Suppose that $\chi_\nu(b) > \chi_M$. Let E^{uu} be the distribution spanned by the Lyapunov subspaces of $(\tilde{\alpha}(b), \nu)$ corresponding to Lyapunov exponents greater than $\chi_M + \varepsilon$. Then, for some $\varepsilon > 0$, E^{uu} has nonzero intersection with the distribution E. It is known that $E^{uu}(x)$ is tangent for ν-a.e. x to the corresponding strong unstable manifold $W^{uu}(x)$. Hence the intersection $F(x)$ of $W^{uu}(x)$ with the leaf $W(x)$ of the coarse Lyapunov foliation corresponding to E is a submanifold of positive dimension. Take $y \in F(x)$ and denote $y_n = \tilde{\alpha}(-nb)(y)$ and $x_n = \tilde{\alpha}(-nb)(x)$. Then x_n and y_n converge exponentially with the rate at least $\chi_M + \varepsilon$. Since the conjugacy $\tilde{\phi}$ is γ bi-Hölder it is easy to see that

$$\text{dist}(\tilde{\phi}(x_n), \tilde{\phi}(y_n)) = \text{dist}(\tilde{\rho}(-nb)(x), \tilde{\rho}(-nb)(y))$$

decreases at a rate faster than $\gamma \chi_M$. But this is impossible since $\tilde{\phi}$ maps $W(x)$ to the corresponding linear foliation which is contracted by $\tilde{\rho}(-b)$ at a rate at most $\gamma \chi_M$. $\diamond$

Proposition 4.2. *Let $L \subset \mathbb{R}^k$ be a Lyapunov hyperplane and E be the corresponding coarse Lyapunov distribution for $\tilde{\alpha}$. For any $\varepsilon > 0$ there exist $C, \eta > 0$ such that for any element $b \in \mathbb{R}^k$ with $\text{dist}(b, L) \leq \eta\varepsilon$, any vector $v \in E$ and any $n > 0$*

$$(2) \qquad\qquad (Ce^{\varepsilon n})^{-1}\|v\| \leq \|D(\tilde{\alpha}(b))v\| \leq Ce^{\varepsilon n}\|v\|.$$

Proof: In the proof we will abbreviate $\tilde{\alpha}(b)$ to b. Consider functions $a_n(x) = \log\|Db^n|_E(x)\|$, $n \in \mathbb{N}$. Since the distribution E is continuous, so are the functions a_n. The sequence a_n is subadditive, i.e. $a_{n+k}(x) \leq a_n(b^k(x)) + a_k(x)$. The Subadditive and Multiplicative Ergodic Theorems imply that for every b-invariant ergodic measure ν the limit $\lim_{n\to\infty} a_n(x)/n$ exists for ν-a.e. x and equals the largest Lyapunov exponent of (b, ν) on the distribution E.

The largest exponent $\bar{\chi}(b)$ of $\tilde{\rho}(b)$ from Lemma 4.1 can be estimated from above by $c \cdot \text{dist}(b, L)$ for some $c > 0$. Hence we can find $\eta > 0$ so that the number χ_M from Lemma 4.1 is less than $\varepsilon/2$ for all $b \in \mathbb{R}^k$ with $\text{dist}(b, L) \leq \eta\varepsilon$. Then Lemma 4.1 implies that $\lim_{n\to\infty} a_n(x)/n \leq \varepsilon/2$ for almost every x with respect to any b-invariant ergodic measure ν. Thus the exponential growth rate of $\|Db^n|_E(x)\|$ is

less than $\varepsilon/2$ for all b-invariant ergodic measures. Since $\|Db^n|_E(x)\|$ is continuous, this is known to imply the uniform exponential growth estimate, as in the second inequality in (2) (see [26, Theorem 1] or [25, Proposition 3.4]). The first inequality in (2) can be obtained from the second one for $-b$. ◇

5. Smooth holonomies.

We consider the suspension actions $\tilde{\alpha}$ and $\tilde{\rho}$ of $\mathbb{R}^k$ on $\mathcal{S}$. We fix a Lyapunov hyperplane $L \subset \mathbb{R}^k$ and denote by E and $\mathcal{W}$ the corresponding coarse Lyapunov distribution and foliation for $\tilde{\alpha}$ on $\mathcal{S}$. In this section we establish smoothness of certain holonomies between leaves of $\mathcal{W}$.

The TNS assumption implies the existence of $\tilde{\alpha}$-invariant distributions E_1 and E_2 such that $E_1 \oplus E \oplus E_2 \oplus T\mathcal{O} = T\mathcal{S}$. Moreover, both E_i and $E_i \oplus E$, $i = 1, 2$ are the stable distribution of some elements, and hence are tangent to invariant foliations which we denote respectively by $\mathcal{W}_i$ and $\mathcal{W}_i \oplus \mathcal{W}$, $i = 1, 2$. To see this consider a generic plane P in $\mathbb{R}^k$ which intersects different Lyapunov hyperplanes by different lines. We can order these oriented lines (i.e. corresponding negative Lyapunov half-spaces) cyclically $L = L_1, L_2, ..., L_n$. Recall that the TNS assumption implies that different negative Lyapunov half-spaces correspond to different Lyapunov hyperplanes. Let m be the index such that $-L_1$ is between L_m and L_{m+1}. There are two Weyl chambers in the negative Lyapunov half-space L_1^- whose intersections with the plane P border L_1. By assumption, there exist Anosov elements in these Weyl chambers, which we denote a_1 and a_2. Similarly, there are two Weyl chambers across L_1 in the positive Lyapunov half-space L^+. We denote Anosov elements in these Weyl chambers by c_1 and c_2. Or, if we order the Weyl chambers intersecting P cyclically from L_1: $\mathcal{C}_i$, $i = 1, ..., n$ then we can take $a_1 \in \mathcal{C}_1$, $a_2 \in \mathcal{C}_m$, $c_1 \in \mathcal{C}_n$, $c_2 \in \mathcal{C}_{m+1}$. If we denote the coarse Lyapunov distribution corresponding to L_i by E^i, then one can see that $T\mathcal{S} = E_1 \oplus E \oplus E_2 \oplus T\mathcal{O}$, where

$$E_2 := E_{c_2}^s = E^2 \oplus .... \oplus E^m \qquad E_{a_2}^s = E^1 \oplus .... \oplus E^m = E \oplus E_2 \qquad \text{and}$$

$$E_1 := E_{c_1}^s = E^{m+1} \oplus .... \oplus E^n \qquad E_{a_1}^s = E^{m+1} \oplus .... \oplus E^n \oplus E^1 = E_1 \oplus E.$$

We will show that the holonomies along $\mathcal{W}_i$, $i = 1, 2$ between leaves of $\mathcal{W}$ are C^∞. This follows from the existence of an element which contracts $\mathcal{W}_1$ (resp. $\mathcal{W}_2$) faster than it does $\mathcal{W}$.

Proposition 5.1. *In the above notations, for $i = 1, 2$, there exist elements $b_i \in \mathbb{R}^k$ such that b_i contracts $\mathcal{W}_i$ faster than it does $\mathcal{W}$, i.e.*

$$(3) \qquad \|D(\tilde{\alpha}(b_i))|_{E_i}\| < \|D(\tilde{\alpha}(-b_i))|_E\|^{-1} \leq \|D(\tilde{\alpha}(b_i))|_E\| < 1.$$

It is known that a faster part of an (un)stable foliation is C^∞ inside an (un)stable leaf, see for example [13, Proposition 5.1] or [14, Proposition 3.9]. Hence we obtain the following corollary:

Corollary 5.2. *In the above notations, for $i = 1, 2$, the leaves of $\mathcal{W}_i$ vary smoothly along the leaves of $\mathcal{W}$, and the holonomies along $\mathcal{W}_i$ between leaves of $\mathcal{W}$ are C^∞.*

Proof: (Of Proposition 5.1.) We consider the case $i = 1$ and denote $a = a_1$, $c = c_1$, and $F = E \oplus E_1$. We have that a uniformly contracts F and c uniformly contracts E_1, i.e. there exist $C_1, \chi > 0$ such that for all $t > 0$

$$(4) \quad \|D(\tilde{\alpha}(ta))v\| \leq C_1 e^{-\chi t}\|v\| \quad \forall v \in F, \qquad \|D(\tilde{\alpha}(tc))v\| \leq C_1 e^{-\chi t}\|v\| \quad \forall v \in E_1$$

Also, there exists the fastest contraction rate χ' for a on E such that for some $c_2 > 0$ and all $t > 0$

$$(5) \qquad\qquad \|D(\tilde{\alpha}(ta))v\| \geq c_2 e^{-\chi' t}\|v\| \quad \forall v \in E$$

Let $b' = ra + (1 - r)c$, $0 < r < 1$, be a convex combination of a and c. Note that by (4) any such b' uniformly contracts E_1:

$$(6) \qquad\qquad \|D(\tilde{\alpha}(tb'))v\| \leq C_1^2 e^{-\chi t}\|v\| \quad \forall v \in E_1, \ \forall t > 0.$$

We will find an element satisfying (3) in the form $b = t(b' + sa)$, where $t > 0$ is large and $s > 0$ is small. For any $\varepsilon > 0$ we can choose b' so that it is in L^- and sufficiently close to L so that Proposition 4.2 applies. Then equations (4), (5), (6) yield that there exists $K > 0$ such that for all $t > 0$

$$\|D(\tilde{\alpha}(b))v\| \leq K e^{-(\chi + s\chi)t}\|v\| \qquad \forall v \in E_1, \qquad\qquad \text{and}$$

$$K^{-1} e^{-(s\chi' + \varepsilon)t}\|v\| \leq \|D(\tilde{\alpha}(b))v\| \leq K e^{-(s\chi - \varepsilon)t}\|v\| \qquad \forall v \in E.$$

We conclude that b will satisfy (3) for sufficiently large t if we choose ε and s so that $s\chi' + \varepsilon < \chi + s\chi$ while $s\chi - \varepsilon > 0$. This is equivalent to

$$\frac{\varepsilon}{\chi} < s < \frac{\chi - \varepsilon}{\chi' - \chi}$$

and hence we can chose such s if ε is sufficiently small. $\diamond$

6. Normal forms

We consider the suspension action $\tilde{\alpha}$ of $\mathbb{R}^k$ on $\mathcal{S}$. We fix a Lyapunov hyperplane $L \subset \mathbb{R}^k$ and denote by E and $\mathcal{W}$ the corresponding coarse Lyapunov distribution and foliation for $\tilde{\alpha}$.

In this section we study properties of the action along the leaves of $\mathcal{W}$ and introduce smooth coordinate changes along the leaves of $\mathcal{W}$ with respect to which the elements act as certain polynomials. This method was introduced to the study of local rigidity of higher rank abelian actions in [19] and uses the nonstationary normal forms of smooth contractions developed in [7, 6]. In contrast to the case of small perturbations of algebraic actions considered in [19], the action $\tilde{\alpha}$ may not have the so-called "narrow band" property. Instead of uniform estimates given by the narrow Mather spectrum, we have to use nonuniform estimates on growth close

to the Lyapunov exponents with respect to μ given by the Multiplicative Ergodic Theorem. Therefore, the coordinate changes will vary on $\mathcal{S}$ not continuously but measurably.

Let a be an element in the negative Lyapunov half-space $L^- \subset \mathbb{R}^k$, so that $f = \tilde{\alpha}(a)$ contracts $\mathcal{W}$. We will view it as a measure-preserving system (f, μ). Its action along $\mathcal{W}$, $f : \mathcal{W}(x) \to \mathcal{W}(fx)$, defines an extension $\Phi : \mathcal{S} \times \mathbb{R}^m \to \mathcal{S} \times \mathbb{R}^m$ of f, where $m = \dim \mathcal{W}$. Indeed, the leaf $\mathcal{W}(x)$ can be smoothly identified with the tangent space $E(x)$, and the distribution E can always be measurably trivialized on a set of full measure. The extension Φ_a preserves the zero section and acts by C^∞ diffeomorphisms in the fibers. In other words, Φ_a can be written in coordinates $(x, t) \in \mathcal{S} \times \mathbb{R}^m$ as

$$\Phi_a(x, t) = (f(x), F_x(t))$$

where $F_x = 0$ and F is C^∞ in t. We will allow coordinate changes which are measurable in x, preserve each fiber $\mathbb{R}^m_x$, fix the origin, are C^∞ in each fiber, and have tempered logarithms of all derivatives of all orders at the zero section. We will call such coordinate changes *admissible*. Recall that a real-valued function φ is called tempered with respect to the action $\tilde{\alpha}$ if $\lim_{b \to \infty} \|b\|^{-1} \varphi(\tilde{\alpha}(b)\, x) = 0$ for μ - a.e. x.

The derivatives in the t variable at the zero section define a linear extension of f, which we will denote by $D_0 F_x$ and call the derivative extension. Note that $D_0 F_x$ are bounded functions on $\mathcal{S}$ and that this extension has negative Lyapunov exponents. Let $\chi_i, \dots \chi_l$ be the different Lyapunov exponents of the derivative extension and $m_1, \dots, m_l$ be their multiplicities. Represent $\mathbb{R}^m$ as the direct sum of the spaces $\mathbb{R}^{m_i}, \dots, \mathbb{R}^{m_l}$ and let $(t_1, \dots, t_l)$ be the corresponding coordinate representation of a vector $t \in \mathbb{R}^m$. Let $P : \mathbb{R}^m \to \mathbb{R}^m$; $(t_1, \dots, t_l) \mapsto (P_1(t_1, \dots, t_l), \dots, P_l(t_1, \dots, t_l))$ be a polynomial map preserving the origin. We will say that the map P is of *subresonance type* if it contains only such homogeneous terms in $P_i(t_1, \dots, t_l)$ with degree of homogeneity s_j in the coordinates of t_j, $i = 1, \dots, l$ for which the subresonance relation $\chi_i \leq \sum_{j \neq i} s_j \chi_j$ holds. There are only finitely many subresonance relations and it is known [6, 7] that polynomial maps of the subresonance type with invertible derivative at the origin generate a finite-dimensional Lie group. We will denote this group by SR_χ. In particular, if there are no resonance relations between the numbers $\chi_1, \dots \chi_l$ then $G_\chi = GL(m, \mathbb{R})$, the group of linear automorphisms of $\mathbb{R}^m$.

Proposition 6.1. *There exists an admissible coordinate change in $\mathcal{S} \times \mathbb{R}^m$ which transforms the extensions Φ_a for all $a \in L^-$ to extensions Ψ_a of the subresonance normal form*

$$\Psi(x, t) = (f(x), \mathcal{P}_x(t))$$

where for almost every $x \in X$, $\mathcal{P}_x \in SR_\chi$.

Moreover, this admissible coordinate change transforms into such normal form any extension $\Gamma(x, t) = (g(x), \mathcal{G}_x(t))$ by C^∞ diffeomorphisms preserving the zero

section of a non-singular transformation g of $(\mathcal{S}, \mu)$ which commutes with Φ_a for some $a \in L^-$.

Proof: We note that since E is a coarse Lyapunov distribution, all Lyapunov exponents of $\tilde{\alpha}$ corresponding to E are, by definition, positively proportional. Therefore, the extensions Φ_a for all $a \in L^-$ are contractions with the same subresonance relations. The existence of an admissible coordinate change for a single $a^* \in L^-$ is given by Theorem 6.1 in [11]. Since Φ_a commutes with Φ_{a^*}, the "centralizer theorem" [11, Theorem 6.3] yields that this coordinate change brings any other Φ_a, for $a \in L^-$, to the subresonance normal form of Φ_{a^*}. The coincidence of resonances implies that this normal form is also the normal form for Φ_a. Then the "centralizer theorem" can be applied to this coordinate change with any $a \in L^-$ and yields the second part of the proposition. ⋄

7. Commuting holonomies

Let $\mathcal{W}$ be a coarse Lyapunov foliation as in the Section 5. In this section, we show that we can put a certain subset of the holonomies described in Section 5 into normal forms coming from Section 6. From now on, we denote the linear conjugate of a foliation or leaf by placing an upper star on the same notation for the object on the non-linear side.

We work with the suspension action and consider an element $v \in \mathbb{R}^k$ which is contracting along $\mathcal{W}$ and which lies in some other Weyl chamber wall. Corresponding to v in the algebraic action there is a foliation $\mathcal{H}_v^*$ along which v acts isometrically and which is defined as the orbits of the action of some subgroup $\mathbf{H}_v^*$ in N. By the TNS assumption, we can always assume that $\mathcal{H}_v^*$ is contained in W_1 (or W_2). Note that $\mathcal{H}_v^*$ is only a full coarse Lyapunov for the action if all Jordan blocks are one dimensional. Also note that the foliation $\mathcal{H}_v^*$ is invariant under the full $\mathbb{R}^k$ (or $\mathbb{Z}^k$) action by definition. In general, $\mathcal{H}_v^*$ corresponds to the subspace of the coarse Lyapunov spanned by the eigenvectors. (While there is a corresponding non-linear foliation $\mathcal{H}_v$, it is not dynamically defined in the presence of Jordan blocks.) Now decompose $\mathbf{H}_v^*$ into the irreducible subspaces $\mathcal{H}_{v,i}^*$ of the rotation defined by v. Then suitable multiples $t_i v$ of v will fix a given $\mathcal{H}_{v,i}^*$. Then the $t_i v$ action on leaves $\mathcal{H}_{v,i}^*$ is in fact trivial. Hence translations by elements of $\mathbf{H}_{v,i}^*$ commutes with $t_i v$.

Note that on the linear side, the holonomies along $\mathcal{H}_{v,i}^*$ between leaves of $\mathcal{W}^*$ are identical to the restriction of elements of $\mathbf{H}_{v,i}^*$ to the leaves of $\mathcal{W}^*$. Therefore if we define the group action of $\mathbf{H}_{v,i}$ on the non-linear side by conjugating by our continuous conjugacy, we have the same statement on the non-linear side : translation by elements in $\mathbf{H}_{v,i}$ restricted to leaves of the corresponding $\mathcal{W}$ agrees with the map between leaves defined by holonomy along $\mathcal{W}_1$. We will consider holonomies of a leaf W in $\mathcal{W}$ along $\mathcal{H}_v$ for different choices of v and i and call any such holonomy a *commuting holonomy* for $\mathcal{W}$.

In particular, this immediately yields the following:

Corollary 7.1. *The action of $\mathbf{H}_{v,i}$ restricted to leaves of $\mathcal{W}$ is smooth.*

The following lemma is immediate from the definitions.

Lemma 7.2. *The elements of $\mathbf{H}_{v,i}$ preserve the normal form along $\mathcal{W}$ for the v action.*

For irreducible actions this lemma would suffice for our purposes, as the action of both $\mathbf{H}_{v,i}$ and $\mathbf{H}_{v.i}^*$ are ergodic.

In more general settings we will need to put larger groups into normal forms. Assume the action is semisimple. As mentioned above, we have that $T\mathcal{W} \oplus T\mathcal{W}_1$ is the full stable for some Anosov element on the linear side. Note that in this setting, there is a group $\mathbf{W}_1$ whose orbits are exactly the leaves of $\mathcal{W}_1$. By making different choices of v and i, we can arrange so that the groups $\mathbf{H}_{v,i}$ generate $\mathbf{W}_1$. The following lemma is immediate from Proposition 6.1. In particular, the last paragraph of that proposition allows us to see that all elements of $\mathbf{W}_1$ share the same normal form.

Lemma 7.3. *If the action is semisimple, then elements of $\mathbf{W}_1^*$ are all smooth along leaves of $\mathcal{W}$ and all preserve, almost everywhere, a fixed normal form along leaves of $\mathcal{W}$.*

8. Limiting Argument

Consider a coarse Lyapunov foliation $\mathcal{W}$. For any leaf W of $\mathcal{W}$ we consider the transitive group G acting on W which is obtained by conjugating translations from the linear side. The main point of this section is to prove the following.

Proposition 8.1. *For any leaf W of $\mathcal{W}$, G acts smoothly on W.*

Proof: It suffices to prove this for some leaf W as we can move any leaf of $\mathcal{W}$ to any other leaf by smooth holonomies coming from the transverse coarse Lyapunov foliations.

Let Λ_m^* be an increasing sequence of Lusin sets for the measurable normal forms whose measures tend to 1, and Λ_m be its density points. Then $\mu(\Lambda_m) \to 1$. Then we can pick a leaf W of $\mathcal{W}$ such that the union of the $W \cap \Lambda_m$ has full measure in W with respect to the conditional measure of μ on W. This can be done as our measure is just the pull back of Lebesgue measure on the linear side.

Fix x and y in $\Lambda_m \cap L$. Since y is a density point of Λ_m we can pick a sequence $x_n \to y$, $x_n \in \Lambda_m$, such that x_n can be reached by commuting holonomies h_n from x. We choose the h_n to lie in the group $\mathbf{W}_1$. Note that $\mathcal{W}_1$ is the stable foliation of an Anosov element, and hence is uniquely ergodic by Bowen and Marcus [2] (alternately one can use algebraic arguments based on work of Auslander, Hahn and Green [1]). Hence we can find the desired elements h_n.

Each h_n is smooth and preserves the normal forms at x and x_n. We may assume that the h_n converge to a homeomorphism $h_{x,y} : W \mapsto W$ since on the linear side they converge to the corresponding translation. Since the normal form coordinates depend continuously on the x_n, and the h_n in these coordinates belong to a fixed Lie group, the limit $h_{x,y}$ is smooth. Hence almost every element in G acts smoothly, and we are done by the following lemma and [22, Section 5.1, Corollary].

Lemma 8.1. *Let G be a Lie group. Then any subgroup H of full measure is G.*

Proof: If not then the distinct cosets of H in G are disjoint sets of full measure which is impossible. □

Remark: It is possible to prove that G is smooth along a generic leaf of $\mathcal{W}$ using older methods involving returns along Weyl chamber walls instead of holonomies. However, one cannot obtain uniformity in estimates this way nor complete the proof below without using holonomies.

End of Proof of Theorem 1.1: Since ϕ intertwines two transitive C^∞-group actions on W and its linear analogue W^*, ϕ is smooth along W. We note that all derivatives of ϕ along leaves of $\mathcal{W}$ are continuous on M. This follows from the fact that $TM = T\mathcal{W} \oplus T\mathcal{W}_1 \oplus T\mathcal{W}_2$ and that the holonomies between different leaves of $\mathcal{W}$ along $\mathcal{W}_1$ and $\mathcal{W}_2$, are smooth and intertwine the restriction of ϕ to these leaves. Now we conclude that ϕ is smooth on M by building a standard elliptic operator and using standard arguments from elliptic operator theory. See e.g. [4, Section 7.1] for a more detailed discussion of this elliptic theory argument.

9. TOTALLY REDUCIBLE ACTIONS AND EXAMPLES.

Here we will prove Corollary 1.2. By the proposition below, this is immediate from Theorem 1.1.

Recall that a linear $\mathbb{Z}^k$ action on a torus is called *irreducible* if there is no rational invariant subtorus, and *totally reducible* if every rational invariant subtorus has a rational invariant complement.

Given a nilmanifold N/Γ, there is a maximal toral quotient $\mathbb{T}^d$ obtained by taking $N/[N,N]\Gamma$. Any action by automorphisms on N/Γ descends to an action on $\mathbb{T}^d$, which we refer to as the *maximal toral quotient action*. We say that a linear $\mathbb{Z}^k$ action on N/Γ is *totally reducible* if the maximal toral quotient action is totally irreducible and there is a $\mathbb{Z}^k$ invariant complement to $[\mathfrak{n}, \mathfrak{n}]$ in the Lie algebra $\mathfrak{n}$ of N.

For affine actions on nilmanifolds, we call the action *totally irreducible* if the finite index subgroup that acts by automorphisms is totally irreducible.

It is easy to see that semisimple actions are totally irreducible.

Proposition 9.1. *A totally reducible $\mathbb{Z}^k$ action on a nilmanifold is semisimple.*

Proof: First we consider an irreducible torus action. Let A be a toral automorphism, i.e. an integral matrix. The characteristic polynomial of A splits over $\mathbb{Q}$ as

$\prod P_i(X)^{d_i}$. Then the kernel $E(A)$ of $\prod P_i(A)$ is the subspace of eigenspaces of A. As $E(A)$ is the kernel of a rational operator, it is rational.

If a collection A_i of toral automorphisms commute then $E(A_1)$ is invariant under A_2. Consider the restriction B_2 of A_2 to $E(A_1)$ Then $E(B_2)$ is nonempty, and contained in $E(A_1) \cap E(A_2)$. Inductively we see that $\cap E(A_i)$ is not empty. Thus we get a nonempty rational subspace invariant under all A_i. This defines an invariant subtorus unless all A_i are semisimple. Hence irreducible torus actions are semisimple.

Considering irreducible components of a totally reducible torus it follows easily that they are also semisimple.

Finally consider a totally reducible action on a nilmanifold. Then the maximal toral quotient action is totally reducible and hence semisimple. This implies that the action on the invariant complement $\mathbb{R}^d$ to $[\mathfrak{n}, \mathfrak{n}]$ is semisimple. Since joint eigenvectors for $\mathbb{Z}^k$ span $\mathbb{R}^d$, their brackets, which are also eigenvectors span $\mathfrak{n}$. Therefore the action is semisimple. $\diamond$

We briefly describe many examples of totally irreducible Anosov actions on nilmanifolds. These examples are more general variants of examples constructed by Qian in [24]. Let $\mathbb{T}^d$ be a torus with an Anosov linear semisimple $\mathbb{Z}^k$ action. The action lifts to the vector space $\mathbb{R}^d$. Let $N = N^k(\mathbb{R}^d)$ be the k-step free nilpotent Lie group generated by $\mathbb{R}^d$. (It is somewhat more typical to define this at the level of Lie algebras, but the meaning is clear as long as we assume $N^k(\mathbb{R}^d)$ is simply connected.) The $\mathbb{Z}^k$ action on $\mathbb{R}^d$ extends canonically to a $\mathbb{Z}^k$ action on $N^k(\mathbb{R}^d)$ and preserves the obvious rational structure on that group. This implies that we have a well-defined $\mathbb{Z}^k$ action on N/Γ where Γ is a lattice in N.

It is easy to check that generically this construction takes an Anosov $\mathbb{Z}^k$ action on $\mathbb{T}^d$ and lifts it to an Anosov action on $\mathcal{N}/\Gamma$. An Anosov automorphism A of $\mathbb{T}^d$ lifts to an Anosov automorphism of N/Γ as long as no product of length at most k of eigenvalues of A has modulus one. It is straightforward to construct many examples which are also TNS using similar algebraic condition on eigenvalues.

We remark that the hypothesis of Theorem 1.1 are necessary for our argument as there are examples for which the commuting holonomies are not ergodic.

Example 9.1. Take a semisimple Anosov linear action of $\mathbb{Z}^k$ on $\mathbb{T}^d$, we can define an action on $\mathbb{T}^{2d}$ by letting $A \in \mathbb{Z}^k$ act by $A(x, y) = (Ax, Ay + x)$. It is straightforward to check that for examples of this kind, the commuting holonomies are not ergodic.

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