Ratner's Rigidity Theorem For Geometrically Finite Fuchsian Groups

by

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1. Introduction. Ratner's rigidity theorem [Ra1] says that a measurable isomorphism of the horocycle flows of two surfaces of constant negative curvature and finite area induces an isometry of the surfaces. There have been various generalizations to variable negative curvature [FeO] and homogeneous spaces of higher dimension [W1,2,F] as well as further-reaching rigidity results like the classification of factors and joinings [Ra2,3].

In the present paper we study Fuchsian groups Γ with infinite co-area. Let $\mathbb{H}^2$ be the hyperbolic plane. Recall that Γ is called geometrically finite if Γ has a finite-sided polygon as fundamental domain in $\mathbb{H}^2$. Let $L(\Gamma)$ be the limit set of Γ, i.e. the set of cluster points of any orbit of Γ in $\mathbb{H}^2$. For a Fuchsian group Γ, Patterson constructed a finite measure in $L(\Gamma) \subset \partial\mathbb{H}^2 = S^1$ with the property that for all $\gamma \in \Gamma$

$$\gamma^* m = |\gamma'|^{\delta} \cdot m$$

where δ is the Hausdorff dimension of $L(\Gamma)$ (and $\gamma^* m(A) = m(\gamma(A))$) [P,S1,2]. Such a measure is called geometric. For geometrically finite groups there is a unique geometric measure m (up to scaling) [S2]. The support of m is always the limit set $L(\Gamma)$. If Γ is not elementary i.e. if Γ is not a finite extension of an abelian group then m has no atoms. Let $M = {}_{\Gamma}\backslash\mathbb{H}^2$ and SM its unit tangent bundle. As usual, identify $S\mathbb{H}^2$ with $(S^1 \times S^1\text{-diag}) \times \mathbb{R}$. The

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Sullivan measure $\tilde{\mu}$ on $S\mathbb{H}^2$ is the measure on $S\mathbb{H}^2$ locally given by

$$d\tilde{\mu}(x,y,t) = \frac{dm(x)\times dm(y)\times dt}{|x-y|^{2\delta}}$$

where $|x-y|$ is the Euclidean distance of x and y in S^1.

It is easily seen that $\tilde{\mu}$ is invariant by the action of Γ on $S\mathbb{H}^2$ and therefore $\tilde{\mu}$ descends to a measure μ on $\Gamma\backslash S\mathbb{H}^2 = SM$.

For geometrically finite groups, μ has finite mass and μ is invariant and ergodic for the geodesic flow g_t [S2]. Further, the support of μ is the nonwandering set Λ of the geodesic flow on SM.

The conditional measure μ_x on the unstable horocycle $W^u(x)$ for $x \in \Lambda$ expands uniformly under the geodesic flow, more precisely

$$(g_t)^* \mu_x = e^{\delta t} \mu_{g_t x}$$

where δ is the Hausdorff dimension of $L(\Gamma)$ as before. Let $\ell(.,.)$ be the distance function on horocycles induced by the canonical metric on $S\mathbb{H}^2$ invariant under isometries of $\mathbb{H}^2$.

Theorem. Let Γ_1 and Γ_2 be Fuchsian groups in $PSL(2,\mathbb{R})$. Assume that Γ_1 is geometrically finite. Denote by M_i, $i = 1,2$, the orbifolds $\Gamma_i\backslash\mathbb{H}^2$. Suppose SM_2 has positive injectivity radius. Let Λ_1 be the nonwandering set of the geodesic flow of M_1, and denote by μ_1 the Sullivan measure on SM_1. Let $\phi: \Lambda_1 \longrightarrow SM_2$ be a measurable map with the properties

a) for μ_1 - a.e.x, $\phi(W^u(x)\cap\Lambda_1) \subset W^u(\phi(x))$.

a) for μ_1 - a.e.x and μ_x - a.e.y on $W^u(x)$,

$$\ell(\phi(x),\phi(y)) = \ell(x,y).$$

Then Γ_1 is conjugate to a subgroup of Γ_2 in the isometry group of $\mathbb{H}^2$. Moreover, after a constant shift along the horocycle ϕ becomes a Riemannian covering map.

In general, Γ_1 may have infinite index in Γ_2. However, there is one important special case.

<u>Corollary</u>. In addition to the hypotheses of the Theorem assume that Γ_2 is geometrically finite and that the Hausdorff dimensions δ_i of the limit sets $L(\Gamma_i)$ coincide. Then Γ_1 is a subgroup of finite index of Γ_2. Furthermore, after a constant translation along the horocycle ϕ becomes a finite Riemannian covering.

The theorem and its corollary have a generalization to geometrically finite groups of isometries of hyperbolic n-space. The proofs are much more technical and will appear elsewhere [FS]. In fact, for simplicity of exposition in the present paper we will always make the additional assumption that Γ_1 is convex cocompact [S1]. This is equivalent to Λ_1 being compact.

The proof of the theorem follows the argument in [Ra1] quite closely. The key ingredient of the proof is the polynomial divergence of orbits just as in the cocompact case [Ra1]. However, we need to take great care in order to control the singular nature of the Sullivan measures. In fact, we were quite surprised to see how robust the basic structure of the proof is. In the cocompact case the conditional measures on horocycles are simply Lebesgue measures which scale trivially under dilation. The main difference in the present case is that the conditionals of Sullivan measures do not enjoy this property. Therefore, we need to use the geodesic flow to "scale" these measures.

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<u>2. Conditional Measures</u>

Here we describe the conditional measures in greater detail and prove some basic properties.

Let Γ be convex cocompact and Λ be the nonwandering set of the geodesic flow g_t on $_\Gamma\backslash S\mathbb{H}^2$. Denote by $\tilde{\Lambda}$ and $\tilde{\mu}$ the lifts of Λ and μ to $S\mathbb{H}^2$. For $x \in S\mathbb{H}^2$ let $\tilde{\mu}_x$ be the measure on $W^u(x)$ which projects to $\mu_{\Gamma \cdot x}$.

If $x \in S\mathbb{H}^2$ let $P(x)$ denote the point at infinity of the geodesic ray determined by x. Then the restriction of P to $W^u(x)$ is a diffeomorphism onto $S^1 - \{P(-x)\}$. Endow $W^u(x)$ and S^1 with their usual Riemannian length. Then we have for $x \in \tilde{\Lambda}$ and $y \in W^u(x)$

(1) $$d\tilde{\mu}_x(y) = \frac{dm(P(y))}{|P'|^\delta}.$$

Let h_t be the classical horocycle flow on $S\mathbb{H}^2$ [Ra1]. For $x \in \tilde{\Lambda}$ identify $W^u(x)$ with $\mathbb{R}$ by

$$t \in \mathbb{R} \longmapsto h_t x \in W^u(x).$$

By abuse of notation, let $\tilde{\mu}_x$ be the measure on $\mathbb{R}$ induced by $\tilde{\mu}_x$ on $W^u(x)$.

<u>Lemma 1</u>. The map $\tilde{\Lambda} \times \mathbb{R} \cup \{-\infty\} \times \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}$ given by

$$(x,a,b) \longmapsto \tilde{\mu}_x(\,]a,b[\,)$$

is continuous on $\tilde{\Lambda} \times \mathbb{R} \times \mathbb{R}$.

<u>Proof</u>. Let $x_n \longrightarrow x$, $a_n \longrightarrow a$ and $b_n \longrightarrow b$ as $n \longrightarrow \infty$ where $a_n, b_n, a, b \in \mathbb{R}$ and $x_n, x \in \tilde{\Lambda}$. Then $P(\,]a_n,b_n[\,) \,\Delta\, P(\,]a,b[\,)$ is contained in arbitrarily small neighborhoods of $\{P(a), P(b)\}$. Since m does not have atoms and P' is uniformly continuous on compact sets, the claim follows from formula (1). ∎

<u>Lemma 2</u>. For $x \in \tilde{\Lambda}$, $\tilde{\mu}_x$ has infinite mass.

<u>Proof</u>. Suppose that $\tilde{\mu}_x(W^u(x)) < \infty$. Since Γ is convex cocompact there are sequences $\gamma_k \in \Gamma$ such that $\gamma_k \cdot g_{-k} x$ converges to

$y \in \tilde{\Lambda}$ as $k \longrightarrow \infty$. Then $\tilde{\mu}_{\gamma_k \cdot g_{-k}x}$ converges to $\tilde{\mu}_y$. Note that

$$\tilde{\mu}_{\gamma_k \cdot g_{-k}x}(W^u(\gamma_k \cdot g_{-k}x)) = \tilde{\mu}_{g_{-k}}(W^u(g_{-k}x)) = e^{-\delta k}\,\tilde{\mu}_x(W^u(x))$$

converges to 0. By Lemma 1, for every $0 < R < \infty$

$$\tilde{\mu}_y(]-R,R[) = \lim_{k\to\infty} \tilde{\mu}_{\gamma_k \cdot g_{-k}x}(]-R,R[).$$

Thus $W^u(y)$ has 0 mass. This is a contradiction since by formula (1) no horosphere at a point in $\tilde{\Lambda}$ has 0 mass. ∎

<u>Lemma 3</u>. For all $x \in \tilde{\Lambda}$ we have

(a) for all $-\infty < a < \infty$,

$$\liminf_{\substack{y\to x,\, y\in\tilde{\Lambda}\\ a'\to a\\ b'\to\infty}} \tilde{\mu}_y(]a',b'[) \geq \tilde{\mu}_x(]a,\infty[)$$

(b) for all $-\infty < b < \infty$,

$$\liminf_{\substack{y\to x,\, y\in\tilde{\Lambda}\\ a'\to-\infty\\ b'\to b}} \tilde{\mu}_y(]a',b'[) \geq \tilde{\mu}_x(]-\infty,b[).$$

(c)
$$\lim_{\substack{y\to x,\, y\in\tilde{\Lambda}\\ a\to-\infty\\ b\to\infty}} \tilde{\mu}_y(]a,b[) = \tilde{\mu}_x(]-\infty,\infty[) = \infty.$$

<u>Proof</u>. (a) Set $A = \tilde{\mu}_x(]a,-\infty[)$.

First assume $A < \infty$. Let $\varepsilon > 0$. Pick N so large that $\tilde{\mu}_x(]a,N[) \geq A - \varepsilon$. By Lemma 1 there are neighborhoods U of x in $\tilde{\Lambda}$ and V of a such that for all $y \in U$, $a' \in V$ and all $n > N$

$$\tilde{\mu}_y(]a',n[) \geq A - 2\varepsilon.$$

This is equivalent to the claim.

Now suppose $A = \infty$. Let $M > 0$. Pick N so large that $\tilde{\mu}_x(]a,N[) \geq M$. Again there is a neighborhood U of x in $\tilde{\Lambda}$ and V of a such that for all $y \in U$, $a' \in V$ and all $n > N$

$$\tilde{\mu}_y(]a',n[) \geq M/2.$$

Again this implies the claim.

(b) The proof is similar to that of (a).

(c) By Lemma 2, since $x \in \tilde{\Lambda}$, $\tilde{\mu}_x(]-\infty,\infty[) = \infty$.

Given $M > 0$ pick N so large that

$$\tilde{\mu}_x(]-N,N[) > M.$$

By Lemma 1 there is a neighborhood U of x such that for all $y \in U$,

$$\tilde{\mu}_y(]-N,N[) > \frac{M}{2}. \quad \blacksquare$$

<u>Lemma 4</u>. Fix $0 < \alpha < 1$. Suppose we have sequences $\{x_n\} \subset \tilde{\Lambda}$, and $-\infty < a_n < b_n < \infty$ with the properties

(a) $a_n < -\frac{\alpha}{2}(b_n - a_n)$ and $\frac{\alpha}{2}(b_n - a_n) < b_n$.

(b) $\tilde{\mu}_{x_n}(]a_n, b_n[) = 1$.

If (x,a,b) is a limit point of $\{(x_n, a_n, b_n)\}$ then $-\infty < a$, $b < \infty$ and $\tilde{\mu}_x(]a,b[) = 1$.

<u>Proof</u>. If $a = -\infty$ then $b = \infty$ and vice versa by (a). Then, by Lemma 3(c), after passing to a subsequence, we have

$$\infty = \tilde{\mu}_x(]-\infty,\infty[) = \lim_{n\to\infty} \tilde{\mu}_{x_n}(]a_n, b_n[) = 1.$$

Therefore a and b are finite, and the last statement follows from Lemma 1. $\blacksquare$

<u>3. Polynomial Divergence of Horocycles</u>

As before, denote by h_s the classical unstable horocycle flow on $S\mathbb{H}^2$ [Ra1]. Recall that $h_s(x)$ is given by polynomials in s and x. Let d be the distance function determined by the canonical metric on $S\mathbb{H}^2$. There are universal constants $1 > \rho > 0$, $C_1 > 1$ and n_0 such that for all $x, y \in S\mathbb{H}^2$ and all intervals $I \subset \mathbb{R}$ on which

$$d^2(h_s x, h_s y) < \rho$$

there exists a polynomial Q of degree at most n_0 such that

$$\frac{1}{C_1} Q(s) \le d^2(h_s x, h_s y) \le C_1 \, Q(s) \qquad \text{for all} \quad s \in I.$$

In the next section we will use the following lemma in combination with the above to estimate the average distance between horocycles.

<u>Lemma 5</u>. There is a constant $C_2 = C_2(\Gamma, n_0) < 1$ with the following property:

Let $a < b$ in $\mathbb{R}$ and $x \in \tilde{\Lambda}$. If Q is any nonnegative polynomial on $[a,b]$ of degree at most n_0 such that
$Q(a) = Q(b) = \sup_{a \le t \le b} Q(t) = M$, then

$$\int_a^b Q(t) d\tilde{\mu}_x(t) \ge \tilde{\mu}_x([a,b]) \cdot M \cdot C_2 .$$

<u>Proof</u>. Without loss of generality we may assume that

$$(*) \quad \sup_{t \in [a,b]} |Q(t)| = Q(a) = Q(b) = 1 \quad \text{and} \quad \deg Q \le n_0 .$$

Also, we may suppose that $\tilde{\mu}\,[a,b] \ne 0$.

Since the set of polynomials satisfying (*) is compact, there exists $\beta < 1$ such that for all polynomials Q satisfying (*) we have

$$Q(t) > \frac{1}{2} \quad \text{for all} \quad t \notin I_\beta(a,b) := \left[\frac{a+b}{2} - \beta\left[\frac{b-a}{2}\right], \frac{a+b}{2} + \beta\left[\frac{b-a}{2}\right]\right].$$

By a scaling argument, β is independent of a and b.

Now we argue by contradiction. Suppose there exist sequences $\{x_n\} \subset \tilde{\Lambda}$, $[a_n, b_n] \subset \mathbb{R}$ and polynomials Q_n, nonnegative on $[a_n, b_n]$ and satisfying (*) on $[a_n, b_n]$ such that

$$(**) \quad \tilde{\mu}_x[a_n, b_n] \ne 0 .$$

and

$$(***)\quad \int_{a_n}^{b_n} Q_n(t)d\tilde{\mu}_{x_n}(t) < \frac{1}{2^n}\,\tilde{\mu}_x([a_n,b_n]).$$

By (*), (**) and (***) we see that $\tilde{\mu}_{x_n}(I_\beta(a_n,b_n)) \neq 0$. Since the support of $\tilde{\mu}$ is $\tilde{\Lambda}$ there are $\tau_n \in I_\beta(a_n,b_n)$ such that $h_{\tau_n}x_n \in \tilde{\Lambda}$. Therefore, replacing x_n by $h_{\tau_n}x_n$, $Q_n(t)$ by $Q_n(t-\tau_n)$, a_n by $a_n-\tau_n$ and b_n by $b_n-\tau_n$, we may assume, in addition, that

$$a_n < -\beta\,\frac{b_n-a_n}{2}$$

and

$$b_n > \beta\,\frac{b_n-a_n}{2}\ .$$

We now proceed to normalize the measures of $[a_n,b_n]$. Set

$$\sigma_n = -\frac{1}{\delta}\log\tilde{\mu}_{x_n}[a_n,b_n]$$

and

$$y_n = g_{\sigma_n}x_n$$

where δ is the Hausdorff dimension of $L(\Gamma)$.

We have

$$\tilde{\mu}_{y_n}\left(\left[a_ne^{\sigma_n},b_ne^{\sigma_n}\right]\right) = e^{\sigma_n\delta}\tilde{\mu}_{x_n}\left(\left[a_n,b_n\right]\right) = 1.$$

Since Γ is convex cocompact after passing to a subsequence of $\{y_n\}$ there are $\gamma_n \in \Gamma$ such that $y:=\lim_{n\to\infty}\gamma_ny_n$ is in $\tilde{\Lambda}$. Note that the sequences γ_ny_n, $c_n:=a_ne^{\sigma_n}$ and $d_n:=b_ne^{\sigma_n}$ satisfy the hypotheses of Lemma 4. Therefore, passing to further subsequences, we may assume that $\lim_{n\to\infty}c_n = c > -\infty$, $\lim_{n\to\infty}d_n = d < \infty$ and $\tilde{\mu}_y([c,d]) = 1$.

Set $\hat{Q}_n(t) = Q_n(t\cdot e^{-\sigma_n})$. The polynomials $\hat{Q}_n(t)$ satisfy

$$\int_{c_n}^{d_n}\hat{Q}_n(t)\,d\tilde{\mu}_{y_n}(t) = \frac{1}{\tilde{\mu}_{x_n}([a_n,b_n])}\int_{a_n}^{b_n}Q_n(t)d\tilde{\mu}_{x_n}(t)$$

since the geodesic flow expands the conditional measures on horocycles uniformly. Therefore, by (***)

$$\int_{c_n}^{d_n} \hat{Q}_n(t)\, d\tilde{\mu}_{\gamma_n y_n}(t) \quad \text{tends to } 0 \text{ as } n \longrightarrow \infty.$$

Passing to a subsequence, $\hat{Q}_n$ converge to a polynomial Q which is nonnegative on $[c,d]$ and for which $Q(c) = Q(d) = 1$. Since c and d are finite the measures $\tilde{\mu}_{\gamma_n y_n}$ converge to $\tilde{\mu}_y$. Therefore

$$\int_a^b Q(t)\, d\tilde{\mu}_y(t) = 0.$$

Since $Q \neq 0$ and $\tilde{\mu}_y$ is not atomic this is a contradiction. ∎

4. Proof of the Theorem. Part I.

In this section we investigate how a map between horocycle foliations intertwines with the geodesic flows. We refer to the statement of the Theorem for all notations, and assume in addition that Γ_1 and Γ_2 are two convex cocompact Fuchsian groups in $PSL(2,\mathbb{R})$.

Note that M_1 and M_2 carry horocycle flows since they are orientable. By [Ru, Theorem 17] the horocycle foliation of M_1 is ergodic for μ_1. Therefore ϕ either preserves the orientations of a.e. horocycle or reverses them. Conjugating Γ_2 by $a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if necessary we may assume that ϕ preserves the orientation of a.e. horocycle. By abuse of notation we will denote the horocycle flows on SM_i (as well as on $S\mathbb{H}^2$) by h_t.

Modifying ϕ on a nullset, if necessary, we may assume that for all $x \in \Lambda_1$, $\phi|_{W^u(x)}$ is an isometry onto its image (since there are no closed horocycles).

Recall the universal constants and ρ and C_1 introduced in the beginning of Section 3. Let C_2 be the constant of Lemma 5 for Γ_1. Denote by R the injectivity radius of M_2. For every $\nu < 1$ set

$\varepsilon = \dfrac{C_2\nu}{8C_1^2}\min(R,\rho)$. Pick $0 < \xi < \varepsilon$. By Lusin's theorem there is a compact set K with μ_1-measure at least $1-\xi$ on which ϕ is uniformly continuous. Choose $\theta > 0$ such that whenever x and y belong to K and $d(x,y) < \theta$ then $d(\phi(x),\phi(y)) \le \frac{\varepsilon}{2}$. Let $\eta > 0$ be a number such that for all $x \in M_i$, $i = 1,2$ and all $|\alpha| < \eta$, $d(x,g_\alpha(x)) < \min(\frac{\varepsilon}{2}, \theta, \frac{\ell n2}{\delta})$. By [Ru] the set

$$\Lambda_1^* = \left\{x \in \Lambda_1 \,\middle|\, \varliminf_{T\to\infty} \frac{1}{\mu_x([-T,T])}\int_{-T}^{T}\chi_K(h_t x)\,d\mu_x(t) > 1-\xi\right\}$$

has full μ_1-measure.

For $x \in \Lambda_1$ and $|\alpha| < \eta$, set

$$P_x(t) = \min\{d^2(h_t\phi(x),\ h_t g_\alpha\phi(g_{-\alpha}x)),\ 1\}.$$

If $x \in \Lambda_1$, we have

1) $\forall t \in \mathbb{R}$ $\qquad h_t\phi(x) = \phi(h_t x)$

2) $\forall t \in \mathbb{R}$ $\qquad h_t g_\alpha\phi(g_{-\alpha}x) = g_\alpha h_{te^{-\alpha}}\phi(g_{-\alpha}x) =$

$$= g_\alpha\phi(h_{te^{-\alpha}}g_{-\alpha}x) =$$

$$= g_\alpha\phi(g_{-\alpha}h_t x).$$

So for $t \in \mathbb{R}$ we have

$$P_x(t) = \min\{d^2(\phi(h_t x),\ g_\alpha\phi(g_{-\alpha}h_t x)),\ 1\}.$$

Whenever $h_t x$ and $g_{-\alpha}h_t x$ both belong to K, our choice of α implies

$$d(\phi(h_t x),\ g_\alpha\phi(g_{-\alpha}h_t x)) \le d(\phi(h_t x),\ \phi(g_{-\alpha}h_t x)) +$$

$$+ d(\phi(g_{-\alpha}h_t x),\ g_\alpha\phi(g_{-\alpha}h_t x)) <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and therefore $P_x(t) < \varepsilon^2$.

From now on, fix $x \in \Lambda_1^* \cap g_\alpha\Lambda_1^*$. We claim that for T_1 and T_2 sufficiently large

$$\mu_x(\{t \in [-T_1, T_2] \mid h_t x \in K \text{ and } g_{-\alpha} h_t x \in K\}) \geq (1-3\xi)\mu_x([-T_1,T_2]).$$

To see this just observe that

$$y \in W^u(x) \longmapsto g_{-\alpha} y \in W^u(g_{-\alpha} x)$$

has Radon-Nikodym derivative $e^{-\delta\alpha} < 2$.

Therefore we obtain for all sufficiently large T_1 and T_2,

$$* \qquad \frac{1}{\mu_x([-T_1,T_2])} \int_{-T_1}^{T_2} P_x(t)\, d\mu_x(t) \leq \varepsilon^2 + 3\xi \leq 4\varepsilon .$$

Using this we are going to show that:

there is $T_0 > 0$ such that

** <u>either</u> $P_x(t) < \hat{\rho}$ for all $t > T_0$,

<u>or</u> $P_x(t) < \hat{\rho}$ for all $t < -T_0$

where $\hat{\rho} = \frac{\nu}{2} \min(\rho, R)$.

Suppose not. Then for all $T_0 > 0$ there are $T_1 > T_0$ and $T_2 > T_0$ such that property * holds for T_1, T_2 and $P_x(-T_1) > \hat{\rho}$ and $P_x(T_2) > \hat{\rho}$.

Let $[a_i, b_i]$ be the intervals in $[-T_1, T_2]$ where $P_x(t) \leq \hat{\rho}$. For every interval $[a_i, b_i]$ there exists a polynomial $Q_i(t)$ such that for all $t \in [a_i, b_i]$,

$$\frac{1}{C_1} Q_i(t) < P_x(t) \leq C_1\, Q_i(t).$$

Let $[c_{i,j}, d_{i,j}]$ be the subintervals of $[a_i, b_i]$ where $Q_i(t) \leq \hat{\rho}/C_1$. By Lemma 5 the μ_x-average of $Q_i(t)$ on each $[c_{i,j}, d_{i,j}]$ is greater than $C_2 \cdot \hat{\rho}/C_1$. Since $C_2 < 1$ the average of Q_i on $[a_i, b_i]$ is also greater than $C_2 \cdot \hat{\rho}/C_1$, and therefore the average of $P_x(t)$ on $[a_i, b_i]$ is greater than $C_2 \cdot \hat{\rho}/C_1^2$.

Thus,

$$\frac{1}{\mu_x([-T_1,T_2])} \int_{-T_1}^{T_2} P_x(t)\, d\mu_x(t) \geq C_2 \cdot \hat{\rho}/C_1^2$$

since $C_1 > 1$. By our choice of ε and $\hat{\rho}$ this contradicts *, and

** is proved.

Since $\hat{\rho} < R$ there are lifts $\tilde{y}_1$ and $\tilde{y}_2$ of $\phi(x)$ and $g_\alpha \phi(g_{-\alpha} x)$ in $S\mathbb{H}^2$ such that $d(h_t y_1, h_t y_2)$ is bounded by $\hat{\rho}$ either as $t \longrightarrow \infty$ or as $t \longrightarrow -\infty$. Thus $y_2 = h_\tau y_1$ for some $\tau = t(x,\alpha) < \hat{\rho}$, and $g_\alpha \phi(g_{-\alpha} x) = h_{\tau(x,\alpha)} \phi(x)$.

As in [Ra1, Lemma 3.3] we can now conclude.

<u>Proposition 6</u>. There is $\rho \in \mathbb{R}$ such that for μ_1-a.e. x and all $t \in \mathbb{R}$

$$h_\tau \circ \phi \circ g_t(x) = g_t \circ h_\tau \circ \phi(x).$$

<u>Proof of the Theorem. Part II</u>.

By Proposition 6, replacing ϕ by $h_\tau \circ \phi$, we may and will assume from now on that ϕ commutes with both geodesic and unstable horocycle flow on a set Ω of full μ_1-measure. We will first show that ϕ commutes with the stable horocycle flow k_t (μ_1-a.e.). As the proof is similar to that of Part I we will only present an outline of the argument.

Let ν, ε, K, ξ, θ and $\hat{\rho}$ be as in Part I. For $T > 0$ set

$$K_T = \{x \in \Omega \,|\, \text{for all } t_1 > T \text{ and } t_2 > T$$

$$\frac{1}{\mu_x([-t_1, t_2])} \int_{-t_1}^{t_2} \chi_K(h_s x) d\mu_x(s) > 1 - \xi\}.$$

By [Ru, Theorem 17], $\mu_1(K_T) \longrightarrow 1$ as $T \longrightarrow \infty$. Pick $T_0 > 0$ such that $\mu_1(K_{T_0}) > \frac{2}{3}$ and set $K_0 = K_{T_0}$. Set

$$\hat{K} = \left\{x \in \Omega \,\middle|\, \varliminf_{T\to\infty} \frac{1}{T} \int_0^T \chi_{K_0}(g_s x) ds > \frac{2}{3}\right\}.$$

Since g_s is ergodic for μ_1, we have $\mu_1(\hat{K}) = 1$. Denote by μ_x^s the conditional measures on stable horocycles $W^s(x)$ for $x \in \Lambda_1$. Then the set

$$\{x \in \hat{K} \mid k_t x \in \hat{K} \quad \text{for} \quad \mu_x^s\text{-a.e. } t\}$$

has full μ_1-measure.

Let $\bar{\eta}$ be so small that for all $|r| < \bar{\eta}$ the holonomy map from $W^u(x)$ to $W^u(k_r x)$ along the weak stable foliation

$$h_s x \in W^u(x) \longmapsto h_{q(s,r)}(k_r x) \longleftarrow W^u(k_r x)$$

has Jacobian between $\frac{1}{2}$ and 2 for all $s \in [-1,1]$. Further, we assume $\bar{\eta}$ to be so small that for all $x \in M_i$, $i=1,2$, and all $|r| < \bar{\eta}$, $t > 0$ and $s \in [-1,1]$, $d(g_t h_s x, g_t h_{q(s,r)} k_r(x)) < \min(\frac{\varepsilon}{2}, \theta)$.

Suppose that $|r| < \eta$ and both x and $k_r x$ belong to $\hat{K}$. Therefore there is a sequence $s_n \longrightarrow \infty$ such that $g_{s_n} x \in K_0$ and $g_{s_n} x \in K_0$. For all $s > 0$ and $t \in [-e^s, e^s]$ we define

$$P_{x,s}(t) = \min\{d^2(h_t \phi(g_s x), h_t k_{-re^{-s}} \phi(k_{re^{-s}} g_s x)), 1\}.$$

As in Part I, one can show that for all $s_n > \ell n\, T_0$ we have

$$A := \frac{1}{\mu_{g_{s_n} x}\left(\left[-e^{s_n}, e^{s_n}\right]\right)} \int_{-e^{s_n}}^{e^{s_n}} P_{x,s_n}(t)\, d\mu_{g_{s_n} x}(t) \le 4\varepsilon$$

since $g_{s_n} x$ and $g_{s_n} k_r x$ both belong to K_0. On the other hand, let $[a_i, b_i] \subset [-e^{s_n}, e^{s_n}]$ be the intervals where $P_{x,s_n}(t) \le \hat{\rho}$. If neither $-e^{s_n}$ nor e^{s_n} belong to such an interval then it follows as in Part I that $A \ge \frac{C_2 \hat{\rho}}{C_1^2} = 8\varepsilon$. This contradicts the above. Therefore there exist sets I_n consisting of at most two intervals containing one of $-e^s$ or e^s on which $P_{x,s}(t) \le \hat{\rho}$. In the complement of I_n the average of $P_{x,s}(t)$ is again greater than 8ε. Therefore for at least one of the components of I_n, call it J_n, we have

$$\frac{\mu_x(g_{-s_n}J_n)}{\mu_x[-1,1]} = \frac{\mu_{g_x s}(J_n)}{\mu_{g_x s}([-e^s,e^s])} > \frac{1}{4}.$$

Passing to a subsequence we may assume that all $g_{-s_n}J_n$ contain the same endpoint. Since μ_x is not atomic there is an interval J contained in all $g_{-s_n}J_n$. For all $s_n > T_0$ and all $t \in e^{s_n}\cdot J$ we have $P_{x,s_n}(t) \le \hat{\rho}$. A simple calculation shows that if $d(h_t x, h_t y) < a$ on an interval $[-L,L]$ then there is a constant $C(a)$ such that $y = k_{\varepsilon_1} g_{\varepsilon_2} h_{\varepsilon_3} x$ where $\varepsilon_1 \le C(a)$, $\varepsilon_2 \le C(a)/_L$ and $\varepsilon_3 \le C(a)/_{L^2}$. Let b be the midpoint of J and let $2L$ be the length of J. Then

$$h_{be^{s_n}} k_{-re^{-s_n}} \phi(k_{r\cdot e^{-s_n}} g_{s_n} x) = k_{\varepsilon_1} g_{\varepsilon_2} h_{\varepsilon_3} h_{b\cdot e^{s_n}} \phi(g_{s_n} x)$$

where $\varepsilon_1 \le C(\hat{\rho})$, $\varepsilon_2 \le C(\hat{\rho})/_{L\cdot e^{s_n}}$ and $\varepsilon_3 \le C(\hat{\rho})/_{L^2 e^{2s_n}}$. The usual commutation rule implies that $\phi(x) = k_{-r}\phi(k_r x)$ for all $|r| < \eta$. Therefore ϕ commutes with the actions of $PSL(2,\mathbb{R})$ on SM_1 and SM_2 on a set of full μ_1-measure. Clearly this allows us to extend ϕ to a $PSL(2,\mathbb{R})$-equivariant map from SM_1 to SM_2. Lift ϕ to a map $\tilde{\phi}$ of the universal cover $S\mathbb{H}^2 \cong PSL(2,\mathbb{R})$ of SM_i, $i = 1,2$. Then $\tilde{\phi}$ is equivariant with respect to the action by right translations by $PSL(2,\mathbb{R})$ on itself.

Let $g_0 = \tilde{\phi}(1)$. Then

$$g_0\Gamma_1 = \tilde{\phi}(1\cdot\Gamma_1) = \tilde{\phi}(\Gamma_1\cdot 1) \subset \Gamma_2\cdot\tilde{\phi}(1) = \Gamma_2\cdot g_0.$$

This proves the Theorem.

6. Proof of the Corollary

As in the statement of the Corollary assume now that Γ_2 is geometrically finite and the Hausdorff dimensions of $L(\Gamma_i)$, $i = 1,2$, coincide and are equal to δ. By [S1,2, Theorem 1] there exists a unique geometric measure of exponents δ on $L(\Gamma_2)$. Therefore there exists a unique measure (up to scaling) on $\Gamma_2\backslash S\mathbb{H}^2$ whose conditional measures on horocycles expand (contract) uniformly with exponent δ under g_t. This is the Sullivan measure μ_2 on SM_2. Observe that $\phi(\Lambda_1) \subset \Lambda_2$, where Λ_2 is the nonwandering set of the geodesic flow on SM_2. Hence, the support of $\phi_*(\mu_1)$ is contained in Λ_2. As the conditional measures of $\phi_*(\mu_1)$ expand (contract) uniformly with exponent δ, $\phi_*(\mu_1)$ is a constant multiple of μ_2. By [S2, Theorem 3] both μ_1 and μ_2 have finite total mass. Clearly this shows that the covering $\phi\colon SM_1 \longrightarrow SM_2$ is finite. Therefore Γ_1 is conjugate to a subgroup of finite index in Γ_2.

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