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ON ISOSPECTRAL LOCALLY SYMMETRIC SPACES AND A THEOREM OF VON NEUMANN

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1. Introduction. Call two lattices Γ_1 and Γ_2 in a locally compact group G *isospectral* if the representations of G on $L^2(G/\Gamma_1)$ and $L^2(G/\Gamma_2)$ are unitarily equivalent. Here we show that isospectral lattices are quite common in semisimple real algebraic groups $\mathbf{G}$. We construct them using a variation of a technique of Sunada [Su]. As a consequence, we obtain new examples of locally symmetric spaces with isospectral Laplacians. In particular, we get such examples with higher rank. Previously, the only higher-rank isospectral manifolds known were locally reducible. These were Vigneras's examples of quotients of products of hyperbolic spaces [V]. On the other hand, Brooks found irreducible higher-rank locally symmetric spaces with isospectral potentials [Br]. However, the underlying manifolds in his examples happened to be isometric. This was due to Mostow rigidity, since the fundamental groups were isomorphic. In fact, this work was largely motivated by the question of whether the rigidity properties of lattices in higher rank force an isospectral rigidity.

THEOREM 1.1. *Let $\mathbf{G}$ be a noncompact almost simple connected real algebraic group whose complexification is of one of the following types:*

(a) *A_n with $n \geqslant 26$*
(b) *C_n with $n \geqslant 27$*
(c) *B_n or D_n with $n \geqslant 13$*

Then any cocompact lattice in $\mathbf{G}$ contains nonisomorphic isospectral torsion-free subgroups of finite index.

We believe that our rank condition is far from optimal. Note that if we fix a maximal compact subgroup K of $\mathbf{G}$, then the locally symmetric spaces $K\backslash\mathbf{G}/\Gamma_i$ defined by Γ_1 and Γ_2 are manifolds as the Γ_i are torsion-free. Since the spectrum of the Laplacian is given by the spectrum of the Casimir element on the K-fixed vectors of $L^2(\mathbf{G}/\Gamma)$, we immediately obtain the geometric:

COROLLARY 1.2. *Let M be a compact locally symmetric space of the noncompact type. Assume that the isometry group of the universal cover of M is as in Theorem 1.1. Then M is finitely covered by two nonisometric isospectral symmetric spaces.*

This corollary yields new examples even in real rank 1, in particular for hyperbolic space.

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Finally, consider actions of a locally compact group G on a measure space S with a finite invariant measure μ. If $L^2(S, \mu)$ is a countable direct sum of irreducible representations of G, we say that the action has *discrete spectrum.* Call the decomposition of $L^2(S, \mu)$ into irreducibles and their multiplicities the *spectrum* of the action. A well-known theorem of von Neumann asserts that two actions of an abelian locally compact group are measurably conjugate if their spectra are discrete and coincide. This fails for semisimple groups:

THEOREM 1.3. *Let* $\mathbf{G}$ *be as in Theorem* 1.1. *Then* $\mathbf{G}$ *has properly ergodic actions with discrete spectrum which are not measurably conjugate and have the same spectrum.*

The proof of Theorem 1.1 is contained in sections 2, 3, and 4. In section 5 we demonstrate Theorem 1.3.

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2. The construction. First we imitate a general technique of Sunada that allows us to construct isospectral lattices.

Let H be a finite group with subgroups H_1 and H_2. Recall that the triple (H, H_1, H_2) satisfies the *conjugacy condition* if for every conjugacy class $[h]$ for $h \in H$,

$$\#([h] \cap H_1) = \#([h] \cap H_2).$$

THEOREM 2.1. *Suppose that* (H, H_1, H_2) *satisfies the conjugacy condition. Let* Γ *be a cocompact lattice in a connected semisimple real algebraic group* $\mathbf{G}$ *with a surjective homomorphism* $\alpha\colon \Gamma \to H$. *If* Γ_1 *and* Γ_2, *respectively, are the preimages of* H_1 *and* H_2 *under* α, *then* Γ_1 *and* Γ_2 *are isospectral.*

Proof. Let Δ be the kernel of α. Since Δ is normal in Γ, H acts on $\mathbf{G}/\Delta$ and hence on $L^2(\mathbf{G}/\Delta)$ by right translation. Since the actions of $\mathbf{G}$ and Γ commute, $\mathbf{G}$ leaves the space $L^2(\mathbf{G}/\Delta)^H$ of Γ-fixed vectors in $L^2(\mathbf{G}/\Delta)$ invariant. Clearly, the representation of $\mathbf{G}$ on $L^2(\mathbf{G}/\Delta)^H$ is isomorphic to that on $L^2(\mathbf{G}/\Gamma)$, and similarly for Γ_1 and Γ_2.

Decompose $L^2(\mathbf{G}/\Delta) = \bigoplus_{\pi \in \hat{\mathbf{G}}} m_\pi \pi$, where $\hat{\mathbf{G}}$ denotes the unitary dual of $\mathbf{G}$ and m_π is the multiplicity of π in $L^2(\mathbf{G}/\Delta)$. Since Δ is cocompact, all m_π are finite and only countably many m_π are not zero. Since the action of H commutes with the representation of $\mathbf{G}$, H leaves the components $m_\pi \pi$ invariant. Thus it suffices to see that $(m_\pi \pi)^{H_1}$ and $(m_\pi \pi)^{H_2}$ are unitarily equivalent for all $\pi \in \hat{\mathbf{G}}$. Fix $\pi \in \hat{\mathbf{G}}$. Of course, $(m_\pi \pi)^{H_1} = c_1 \pi$ and $(m_\pi \pi)^{H_2} = c_2 \pi$ are again multiples of π. We will show that $c_1 = c_2$ by a trace computation.

Let f be a smooth, compactly supported function on $\mathbf{G}$ and define

$$\pi(f)v \overset{\text{def}}{=} \int_{\mathbf{G}} f(g)(\pi(g)v)\, dv$$

for $v \in m_\pi \pi$. Since $m_\pi \pi$ is admissible, $\pi(f)|m_\pi \pi$ is trace class by a theorem of Harish-Chandra [HC]. Pick f such that $\operatorname{tr} \pi(f)|\pi \neq 0$ (that this is possible follows, for example, from the linear independence of the distribution characters). As in Lemma 2 of [Su], it follows from the elementary trace formula [Su, Lemma 1] that

$$\operatorname{tr} \pi(f)|(m_\pi \pi)^{H_1} = \operatorname{tr} \pi(f)|(m_\pi \pi)^{H_2}.$$

As

$$c_i = \frac{\operatorname{tr} \pi(f)|(m_\pi \pi)^{H_i}}{\operatorname{tr} \pi(f)|\pi},$$

we obtain $c_1 = c_2$. □

3. Some nice triples. Here we introduce the basic triples we will work with. Let $\mathscr{A}_{27}$ be the alternating group on twenty-seven letters, $H_1 = \mathbf{Z}_3^3$ and H_2 the Heisenberg group over $\mathbf{Z}_3$, i.e., the group of strictly upper-triangular 3×3 matrices with coefficients in $\mathbf{Z}_3$. We embed H_1 and H_2 into the permutation group on twenty-seven letters by letting H_1 and H_2 act on themselves by left translation. As H_1 and H_2 have order 3, the values lie in $\mathscr{A}_{27}$.

LEMMA 3.1. *Let H' be any finite group that contains $\mathscr{A}_{27}$ as a subgroup. Then the triple (H', H_1, H_2) satisfies the conjugacy condition.*

Proof. First, notice that the cycle representation of any element of H_1 or H_2 in H consists of nine 3-cycles. It is easy to check that there is only one conjugacy class of elements with nine 3-cycles in $\mathscr{A}_{27}$. Hence $H_1 - \{1\}$ and $H_2 - \{1\}$ are contained in the same conjugacy class of $\mathscr{A}_{27}$ and therefore of H' for any $H' \supset \mathscr{A}_{27}$. □

We obtain many such triples from

LEMMA 3.2. *Let $\mathbf{G}$ be a connected adjoint split simple algebraic group of rank n over a finite field $\mathbf{F}_p$ of cardinality q. Then $\mathbf{G}$ contains $\mathscr{A}_{27}$ provided that*

(a) $n \geqslant 26$ *if $\mathbf{G}$ is of type A_n;*
(b) $n \geqslant 14$ (*resp.* 13) *if $\mathbf{G}$ is of type B_n and $q = 3 \pmod 4$* (*resp.* $q = 1 \pmod 4$));
(c) $n \geqslant 27$ *if $\mathbf{G}$ is of type C_n;*
(d) $n \geqslant 14$ (*resp.* 13) *if $\mathbf{G}$ is of type D_n and $q = 3 \pmod 4$* (*resp.* $q = 1 \pmod 4$)).

Proof. For A_n, we embed $\mathscr{A}_{27}$ into $SL(n+1, \mathbf{F}_p)$ by letting $\mathscr{A}_{27}$ act on the coordinates. Since $\mathscr{A}_{27}$ is simple, $\mathscr{A}_{27}$ embeds into the adjoint group. The proof for the other types is similar. For types B_n and D_n, we use that there are only one or two nondegenerate quadratic forms over $\mathbf{F}_p$, depending on whether $q = 1$ or $3 \pmod 4$, namely, the forms $x_1^2 + \cdots + x_n^2$ or $x_1^2 + \cdots - x_n^2$ [S]. □

4. Lattices. In this section we prove Theorem 1.1. First, we construct maps from lattices onto algebraic groups over finite fields by reduction modulo a prime ideal.

We use a version of strong approximation here. Using Theorem 2.1, we obtain isospectral lattices. To finish, we show that these lattices are not isomorphic.

Let $\mathbf{G}$ be a semisimple connected real algebraic group without compact factors, and let Γ be an irreducible cocompact lattice in $\mathbf{G}$. By a lemma of Selberg [Ra, Theorem 6.11] there is a torsion-free subgroup Γ' of Γ of finite index. Thus we will always assume that Γ is torsion-free. Suppose also that $\mathbf{G}$ is not locally isomorphic to $SL(2, \mathbf{R})$. Then there is a number field K such that $\Gamma \subset \mathbf{G}(K)$ [Ra, Theorem 7.67] and such that $\mathbf{G}$ splits over K [B-T, Theorem 2.14]. Let $\mathbf{G}'$ be the $\mathbf{Q}$-group obtained by restriction of scalars from K to $\mathbf{Q}$. There is a bijection $\sigma = (\sigma_1, \ldots, \sigma_d)$: $\mathbf{G}(K) \to \mathbf{G}'(\mathbf{Q})$ determined by the embeddings σ_i of K into $\mathbf{C}$. In particular, this defines an embedding of Γ into $\mathbf{G}'(\mathbf{Q})$. Since σ is defined over $\mathbf{C}$ and Γ is Zariski-dense in $\mathbf{G}(\mathbf{C})$, by the Borel density theorem Γ is Zariski-dense in $\mathbf{G}'(\mathbf{Q})$.

Thus we will assume from now on that $\mathbf{G}'$ is semisimple and split and that Γ is torsion-free and Zariski-dense in $\mathbf{G}'(\mathbf{Q})$. Since Γ is finitely generated, there is a finite set of primes S such that $\mathbf{G}'$ is defined over $R \overset{\text{def}}{=} S^{-1}\mathbf{Z}$ and such that $\Gamma \subset \mathbf{G}'(R)$. Note that Γ is again Zariski-dense in $\mathbf{G}'(R)$. For all but finitely many primes p, reducing modulo p in R, we obtain the semisimple and split group $\mathbf{G}'(\mathbf{F}_p)$ over the finite field $\mathbf{F}_p$. In particular, $\mathbf{G}'(\mathbf{F}_p)$ is generated by its unipotents. Therefore, for all but finitely many primes p, we can apply Theorem 5.1 of [N] and see that the image of Γ by the reduction map modulo p equals the group $\mathbf{G}'(\mathbf{F}_p)$. We denote the composition of reduction modulo p with the projection to the adjoint group $\mathbf{G}'_{ad}$ by α. Suppose also that $p = 1 \pmod 4$. By Lemma 3.2 there is a triple $(\mathbf{G}'_{ad}(\mathbf{F}_p), H_1, H_2)$ with H_1 and H_2 as in section 3. Set $\Gamma_i \overset{\text{def}}{=} \alpha^{-1}(H_i) \cap \Gamma$. By Sunada's theorem, Γ_1 and Γ_2 are isospectral. As Γ is torsion-free, so are the Γ_i. In the remainder of this section we will show that Γ_1 and Γ_2 are not isomorphic in this case. This will finish the proof of Theorem 1.1.

Let C be the p-congruence subgroup of Γ, i.e., the kernel of α in Γ.

LEMMA 4.1. *The order of $C/[C, C]$ is a power of p.*

Proof. Let C_i denote the ith congruence subgroup, i.e., the kernel in Γ of reduction mod p^i. Note that $[C, C] \subset C_2$. Then the order of

$$C/[C, C] \Big/ C_2/[C, C] \approx C/C_2$$

is a power of p, since C/C_2 is a subgroup of the additive group of $n \times n$ matrices (where $\mathbf{G}_{ad}$ is a subgroup of $GL(n)$). Similarly,

$$C_2/[C, C] \Big/ C_3[C, C]/[C, C] \approx C_2/C_3[C, C]$$

has order a power of p, etc. Since $\bigcap C_i = \{1\}$, the claim follows. □

LEMMA 4.2. *We have the isomorphisms*

(a) $\Gamma_1/[C, C] \approx H_1 \ltimes C/[C, C]$;
(b) $\Gamma_2/[C, C] \approx H_2 \ltimes C/[C, C]$.

Proof. Because $p \neq 3$, the second cohomology $H^2(\mathbf{Z}_3^3, C/[C, C])$ vanishes [Bn, Corollary 10.2]. Hence we see that

$$\Gamma_1/[C, C] \approx \mathbf{Z}_3^3 \rtimes C/[C, C].$$

The second claim follows similarly. □

Finally, we can finish the proof of Theorem 1.1, as explained above, by

LEMMA 4.3. *If p is not 3, then Γ_1 and Γ_2 are not isomorphic.*

Proof. First notice that the abelianization of Γ_1 contains H_1. Indeed,

$$\Gamma_1/[\Gamma_1; \Gamma_1] \approx \Gamma_1/[C, C]/[\Gamma_1, \Gamma_1]/[C, C]$$

is isomorphic with the abelianization of $\mathbf{Z}_3^3 \rtimes C/[C, C]$. Thus $\#(\Gamma_1/[\Gamma_1, \Gamma_1]) = 27p^k$ for some integer k. On the other hand, as above for Γ_1, $\Gamma_2/[\Gamma_2, \Gamma_2]$ is the abelianization of $H_2 \rtimes C/[C, C]$, whose order is not divisible by 27, as H_2 is not abelian. □

5. Isospectral actions of semisimple groups. Here we mainly discuss the proof of Theorem 1.3.

Let $\mathbf{G}$ be as in Theorem 1.1. Without loss of generality, we will also assume that $\mathbf{G}$ is an adjoint group. Borel showed how to construct lattices in $\mathbf{G}$ [B], [Ra]. In fact (cf. the proof of Theorem 14.2 of [Ra]), there is an algebraic $\mathbf{Q}$-group $\mathbf{H}$ with the properties

(a) $\mathbf{H}(\mathbf{R}) = \mathbf{G} \times Ł$, where $Ł$ is a compact real algebraic group;
(b) the projection Γ of π_1 of $\mathbf{H}(\mathbf{Z})$ to $\mathbf{G}$ is a cocompact lattice;
(c) the projections π_1 and π_2 to $\mathbf{G}$ and $Ł$, respectively, are injective on $\mathbf{H}(\mathbf{Z})$.

Let M be the Hausdorff closure of $\pi_2(\mathbf{H}(\mathbf{Z}))$ in $Ł$. Then $\mathbf{H}(\mathbf{Z}) \subset \mathbf{G} \times M$. We will identify Γ with $\mathbf{H}(\mathbf{Z})$ via π_1. Let $S \overset{\text{def}}{=} \mathbf{G} \times M/\Gamma$ and let μ be Haar measure on S. We let $\mathbf{G}$ act on S by left translations.

LEMMA 5.1. *The action of $\mathbf{G}$ on S has discrete spectrum.*

Proof. Let $\hat{\sigma}$ denote the representation of Γ on $L^2(M)$ determined by σ. Then the representation of $\mathbf{G}$ on $L^2(S)$ is the induced representation of $\hat{\sigma}$ from Γ to $\mathbf{G}$. Since the representation of M on $L^2(M)$ is discrete, the lemma follows immediately from [G-G-P, 1:2.3]. □

Let us note here that in general the multiplicities of the irreducible representations of $\mathbf{G}$ in $L^2(S)$ are infinite.

Proof of Theorem 1.3. By the proof of Theorem 1.1, there is a map from Γ into a triple as in Lemma 3.2. Let Γ_1 and Γ_2 be the preimages of H_1 and H_2, respectively, in Γ. Then Γ_1 and Γ_2 are isospectral lattices in $\mathbf{G} \times M$. Since the representations of $\mathbf{G}$ on $L^2(\mathbf{G} \times M/\Gamma_i)$ are the restrictions of the representations of $\mathbf{G} \times M$ on $L^2(\mathbf{G} \times M/\Gamma_i)$, we see that the actions of $\mathbf{G}$ on these spaces are isospectral. It remains to show that the actions of $\mathbf{G}$ on these spaces are not measurably conjugate. First, we recall that the action of $\mathbf{G}$ and therefore of any unipotent group on $\mathbf{G} \times M/\Gamma_i$ is properly ergodic by [Bz-M, Theorem 5.5]. Hence the claim follows from [W, Theorem 2.1′]. □

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