Invariant Measures for Higher Rank Hyperbolic Abelian Actions

A. Katok *
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802

R. J. Spatzier †
Department of Mathematics
University of Michigan
Ann Arbor, MI 48103

Abstract

We investigate invariant ergodic measures for certain partially hyperbolic and Anosov actions of $\mathbb{R}^k$, $\mathbb{Z}^k$ and $\mathbb{Z}^k_+$. We show that they are either Haar measures or that every element of the action has zero metric entropy.

1 Introduction

Actions of higher rank abelian groups and semigroups on compact smooth manifolds display a remarkable and not yet completely understood array of rigidity properties provided the action is sufficiently hyperbolic. Early indications of such phenomena can be found in the works of N. Koppel and R. Sacksteder on commuting one-dimensional and expanding maps [17, 27]. A. Katok and J. Lewis established local and global differential rigidity of the actions of $\mathbb{Z}^{n-1}$ on T^n by hyperbolic toral automorphisms [11]. Some of the phenomena including trivialization of the first cohomology group, absence of non-trivial time changes, local Hölder and differential rigidity for a general class of standard abelian actions are studied in our papers [13, 14, 15]. For related developments see [10, 12].

Another of those rigidity properties is the relative scarcity of invariant Borel probability measures. It was first noticed by H. Furstenberg in his landmark paper [5] where he posed the following problem.

Furstenberg's Conjecture : *The only ergodic invariant measures for the semigroup of circle endomorphisms generated by multiplications by p and q where $p^n \neq q^m$ unless $n = m = 0$ are Lebesgue measure and atomic measures concentrated on periodic orbits.*

Furstenberg establishes the weaker topological version of this statement by showing that all topologically transitive sets are either finite or the whole circle. This was generalized

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to the optimal results for semigroups of toral endomorphisms by D. Berend [2, 3]. E. A. Satayev proved that the only ergodic invariant measures are Lebesgue and atomic for larger semigroups generated by multiplications by $P(n)_{n\in\mathbb{Z}_+}$ where P is any polynomial with integer coefficients [28]. The first result directly pertaining to Furstenberg's conjecture was obtained by R. Lyons using harmonic analysis. He proved that the only invariant measure which makes the multiplications exact endomorphisms is Lebesgue [21]. D. Rudolph and A. Johnson strengthened this result by replacing the exactness condition with positive entropy for some and hence all elements of the action [26, 8]. At the heart of their arguments lies a symbolic version of the natural extension of a $\mathbb{Z}_+^2$-action to a $\mathbb{Z}^2$-action. For further developments in this specific problem see [4, 6].

More generally, one notices a sharp contrast between Anosov diffeomorphisms and flows, i.e. hyperbolic actions of $\mathbb{Z}$ and $\mathbb{R}$, which possess an abundance of invariant measures with very different ergodic properties, including many measures with positive entropy, and "genuine" hyperbolic actions of higher rank abelian groups and semigroups. In the latter case, all known ergodic invariant measures are of algebraic nature unless, like in an example constructed by M. Rees in an unpublished manuscript [24], there is an invariant submanifold on which the action has a factor where it reduces to an action of a rank one group. The question of deciding what hyperbolic (Anosov) or partially hyperbolic actions should be considered "genuine" is rather subtle. Obviously, in addition to faithfulness one should require the absence of rank one factors for the action and all of its finite covers. The central open question in the area is whether all such actions are of algebraic nature (cf. [9]). For the time being, it is reasonable to list all known examples and bundle them together under the name of *standard* actions. These include irreducible semigroups of partially hyperbolic endomorphisms of tori and (infra)nilmanifolds, their natural extensions and suspensions, Weyl chamber flows and related symmetric space examples and twisted Weyl chamber flows (cf. Sections 3 and 6 as well as [13]). Then the central open problem concerning invariant measures can be formulated in the following way.

All standard examples act on biquotients M of a Lie group G. We call a submanifold M' of M *homogeneous* if its preimage in G is a coset of a closed subgroup. We call a measure on a finite union of homogeneous submanifolds *Haar* if its restriction to any of the homogeneous submanifolds can be constructed by projecting Haar measure on a coset in G to M.

Main Conjecture: *Let α be a standard Anosov action of $\mathbb{Z}_+^k$, $\mathbb{Z}^k$ or $\mathbb{R}^k$, $k \geq 2$ on a manifold M. Then any α-invariant ergodic Borel probability measure μ is either Haar measure on a homogeneous real algebraic subspace or the support of μ is a homogeneous subspace M' which fibers in an α-invariant way over a manifold N such that the α-action on N reduces to a rank one action, i.e. the action of $\mathbb{Z}_+$, $\mathbb{Z}$ or $\mathbb{R}$.*

In particular, if the support of μ is all of M then μ is Haar measure on M.

The second alternative includes measures supported on closed orbits of the action. The set of such orbits is always dense. Aside from those measures, the second alternative does not appear in the standard toral examples and appears to be rather exceptional in the symmetric space examples.

A similar conjecture can be stated for a more general class of partially hyperbolic actions where one may have to allow natural measures on some non-homogeneous real algebraic submanifolds.

In this paper we consider invariant ergodic measures for certain homogeneous actions of higher rank abelian groups. Our main assumption is similar to that of Rudolph and Johnson, namely that some element has positive entropy w.r.t. the measure in question. In the most general case, we have to assume more. For $\mathbb{R}^k$-actions for example, it is sufficient to assume that every one-parameter subgroup is ergodic. The similar assumptions for $\mathbb{Z}^k$- and $\mathbb{Z}^k_+$-actions are that one parameter subgroups of the suspension and correspondingly the suspension of the natural extension are ergodic. In particular, all mixing measures satisfy these assumptions. These conditions exclude measures coming from Rees's examples since those measures are not ergodic with respect to certain one-parameter subgroups.

Under those or slightly weaker assumptions, we show in the toral and semisimple (symmetric space) cases that the measure is Haar measure on a homogeneous algebraic subspace (Theorems 5.1 and 7.1; Corollaries 5.2 and 5.3, and analogous statements for the semisimple case). In many cases, where there are no non-trivial homogeneous algebraic invariant subspaces, this implies that the measure is Haar measure on the whole space. In the case of twisted Weyl chamber flows, to achieve similar conclusions we need to assume in addition that the projection to the semisimple factor has positive entropy for some element (Theorem 7.2).

For certain toral actions, essentially the totally non-symplectic actions, the extra assumptions can be removed. Thus we obtain a generalization of the Rudolph and Johnson results which covers certain commuting expanding toral endomorphisms, Anosov actions of higher rank subgroups of $\mathbb{Z}^{n-1}$ on T^n by automorphisms and many other examples (Corollary 6.4).

The main idea of our argument is to decompose the invariant measure into conditionals along stable and unstable foliations of various elements of the action. These foliations are homogeneous. By looking at conditionals at various invariant subfoliations we show that some of those conditional measures are either atomic or Haar along a homogeneous subfoliation. In the first case, the entropy of some and then every element is zero. In the second case, rigidity follows in the toral case from unique ergodicity of a linear flow on the torus on its orbit closures and in the semisimple and twisted cases from M. Ratner's classification of invariant measures for homogenous actions of unipotent groups [23].

Let us point out that our method which is based on the local structure of stable and unstable foliations for various elements breaks down for symplectic actions of $\mathbb{Z}^k$ on even–dimensional tori. Such actions may be totally irreducible (no invariant rational subtori); explicit examples of that kind starting from $\mathbb{Z}^2$ actions on T^4 were shown to us by L.Vaserstein. However, the local structure of such actions as presented by Lyapunov decompositions, Weyl chambers and Lyapunov hyperplanes (see next section) is undistinguishable from that of the products of rank one actions.

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2 Lyapunov exponents

We will study Anosov and, more generally, partially hyperbolic actions of $\mathbb{Z}_+^k$, $\mathbb{Z}^k$ and $\mathbb{R}^k$. For a general discussion of such actions we refer to [15, 14]. As we will see, it is more convenient for our approach to operate with $\mathbb{R}^k$-actions. Therefore let us first explain how to pass from an action of $\mathbb{Z}^k$ to $\mathbb{R}^k$. This is the so-called suspension construction.

Suppose $\mathbb{Z}^k$ acts on N. Embed $\mathbb{Z}^k$ as a lattice in $\mathbb{R}^k$. Let $\mathbb{Z}^k$ act on $\mathbb{R}^k \times N$ by $z(x, m) = (x - z, z.m)$ and form the quotient

$$M = \mathbb{R}^k \times N/\mathbb{Z}^k.$$

Note that the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times N$ by $x.(y, n) = (x + y, n)$ commutes with the $\mathbb{Z}^k$-action and therefore descends to M. This $\mathbb{R}^k$-action is called the *suspension* of the $\mathbb{Z}^k$-action.

Note that any $\mathbb{Z}^k$-invariant measure on N lifts to a unique $\mathbb{R}^k$-invariant measure on the suspension.

Furthermore we can pass from a $\mathbb{Z}_+^k$-action to a $\mathbb{Z}^k$-action by a natural projective limit construction in an appropriate category. This construction is explained in detail for toral endomorphisms in Section 3 where it is called the solenoid construction. As we will see in the appendix, the solenoids are locally modeled on the products of certain p-adic rings of integers with $\mathbb{R}^k$. Let us also mention that any invariant measure on the torus canonically lifts to the solenoid.

A crucial role in our analysis of $\mathbb{R}^k$-actions is played by the Lyapunov exponents. Consider a measure preserving ergodic action of $\mathbb{R}^k$ on a space X. Suppose $\mathbb{R}^k$ acts by bundle automorphisms on a bundle over X with products of real and p-adic vectorspaces as fibers covering the given action on X. For a single element a in the group and a vector v in the extension, the Lyapunov exponent $\lambda(a, v)$ is defined in the usual way (compare with [22, ch. V]). There is a decomposition into Lyapunov subspaces of the extension a.e. such that the different Lyapunov exponents of a are given as Lyapunov exponents of a and some vector in the Lyapunov space. Due to the commutativity of the group, we can find a common refinement of the Lyapunov decompositions of single elements of the group. We will call this refined decomposition the *Lyapunov decomposition* of the extension. This allows us to consider the Lyapunov exponents $\lambda(\ , v)$ for v in a Lyapunov space of the extension as a real valued functional on the group. Since the acting group is abelian, the Lyapunov exponents are linear functionals on the group. A particular example of such an extension for a smooth system is its derivative. We refer to [7] for a more detailed exposition of Lyapunov exponents in this case.

When we speak about Lyapunov exponents of a $\mathbb{Z}^k$-action or a $\mathbb{Z}_+^k$-action we will always mean those for the suspension and correspondingly the suspension of the natural extension of the given action. Consider the finitely many hyperplanes in $\mathbb{R}^k$ defined by the vanishing of the functionals. We will call these hyperplanes the *Lyapunov hyperplanes*. Let us call an element $a \in \mathbb{R}^k$ *regular* if it does not belong to the kernel of any non-trivial functional. All other elements are called *singular*. Call a singular element *generic* if it belongs to only one Lyapunov hyperplane. Note that the tangent space to the $\mathbb{R}^k$-orbit defines the identically 0 Lyapunov exponent. Let us emphasize that Lyapunov exponents may be proportional to each other with positive or negative coefficients. In this case, they define the same Lyapunov hyperplane.

The Lyapunov hyperplanes divide $\mathbb{R}^k$ into finitely many open connected components, called the *Weyl chambers* of the action. Thus every regular element belongs to a unique Weyl chamber. Every generic singular element belongs to the common boundary of exactly two Weyl chambers. The system of Weyl chambers is symmetric w.r.t. the origin. Thus for any Weyl chamber $\mathcal{C}$, $-\mathcal{C}$ is also a Weyl chamber.

Note that the Lyapunov hyperplanes cannot be are not derictly seen from a $\mathbb{Z}^k$ or $\mathbb{Z}^k_+$-action as the hyperplanes are not rational in general. This is one of the reasons making $\mathbb{R}^k$ actions a more convenient object of study.

In all homogeneous examples, standard or not, the Lyapunov exponents for the derivative extension are defined and constant everywhere. In particular, they are independent of the invariant measure. They determine a splitting of the tangent bundle (which may have p-adic components in the solenoid case) into invariant subbundles called the *Lyapunov spaces*. Let us emphasize that the Lyapunov spaces in the p-adic directions correspond to closed subgroups of some $\mathbb{Z}_p^m$. The dimension of each Lyapunov space will be called the *multiplicity* of the exponent (where dimension for a p-adic direction is the dimension of corresponding p-adic modules, c.f. the Appendix). The multiplicity of the 0 exponent is at least k. If the multiplicity of the 0 exponent is exactly k, we call the action *Anosov*. A regular element for an Anosov action is called an *Anosov element*.

For an element $a \in \mathbb{R}^k$ let us define the *stable, unstable* and *neutral* distribution E_a^+, E_a^- and E_a^n as the sum of the Lyapunov spaces for which the value of the corresponding Lyapunov exponent on a is negative, positive and 0 respectively. The neutral distribution for any element of an $\mathbb{R}^k$-action contains the tangent distribution to the orbit; in the non-Anosov partially hyperbolic case it also contains other directions corresponding to the Lyapunov exponents identically equal to 0. We will be interested in the complement to these "trivial" directions in the neutral distribution of a singular element. It is defined as follows.

Note that the stable and unstable distributions are constant on a Weyl chamber. Note furthermore that the sum of stable and unstable distributions is constant for all regular elements. We will denote this sum by E^H. For singular elements the neutral distributions are bigger than for regular ones. For example, for a generic singular element the neutral distribution contains a direction which is stable for one adjacent Weyl chamber and unstable for the other. We will call the intersection of the neutral distribution for an element a with the distribution E^H the *center* distribution of a and denote it by E_a^0. The center distribution for any singular element a always contains directions on which the derivative of a acts isometrically w.r.t. a canonical homogeneous metric. We denote the distribution of isometric directions inside E_a^0 by E_a^I.

Let us note that the Lyapunov spaces in the standard toral examples always integrate to affine foliations (possibly with a p-adic part). In the symmetric space examples, some of the Lyapunov spaces may not be integrable (cf. the discussion in the proof of Theorem 7.1). However, stable, unstable and center distributions as well as their intersections (for different elements) are always integrable and integrate to homogeneous foliations. In fact, the stable and unstable foliations of any element are always the orbit foliations of a unipotent subgroup. For an element a we will denote the integral foliations of the stable, unstable, center and isometric distributions E_a^+, E_a^-, E_a^0 and E_a^I by W_a^+, W_a^-, W_a^0 and W_a^I.

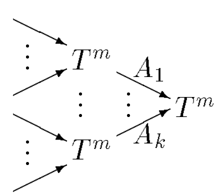
3 Toral endomorphisms, solenoids and their suspensions

Consider an embedding α of $\mathbb{Z}_+^k$ into the semigroup of non-singular $m \times m$ integer matrices. Then $\mathbb{Z}_+^k$ acts on the torus T^m by endomorphisms. Note that this includes actions of $\mathbb{Z}^k$ by toral automorphisms by restricting to $\mathbb{Z}_+^k$. We will always assume that every non-trivial element of $\mathbb{Z}_+^k$ acts ergodically with respect to Haar measure, or equivalently, that no non-trivial element of $\mathbb{Z}_+^k$ has eigenvalues that are roots of unity. Such an action is called *irreducible* if no finite cover splits as a product. Irreducible actions by ergodic toral endomorphisms are called *standard* actions. Furthermore, in agreement with the terminology of the previous section, we will call α *Anosov* if the image of $\mathbb{Z}_+^k$ contains matrices without eigenvalues on the unit circle.

Note that if the actions admits a factor on which the action reduces to an action of $\mathbb{Z}$ or $\mathbb{Z}_+$, then invariant measures cannot be rigid. In this case however, any element in the kernel of the action on the factor has 1 as an eigenvalue. Thus actions by ergodic toral automorphisms do not admit such factors. Conjecturally, the presence of "rank one factors" is the only obstruction to rigidity.

To make these actions invertible we will introduce the natural extension $\alpha^* : \mathbb{Z}^k \to Aut(S)$ of α where S is the solenoid obtained from the torus as follows.

Let $A_1, \ldots A_k$ be the images of the generators of $\mathbb{Z}_+^k$. Then we get a projective system



where the maps are given by the A_i. We let the *solenoid* S be the projective limit of this system in the category of compact topological groups.

The solenoid can be realized as a subset of $(T^m)^{\mathbb{Z}^k}$ as follows. Let σ_i be the i'th shift on $\mathbb{Z}^k$ i.e. $\sigma_i(j_1, \ldots, j_i, \ldots, j_k) = (j_1, \ldots, j_i + 1, \ldots, j_k)$. Then set

$$S = \{\omega \in (T^m)^{\mathbb{Z}^k} \mid \omega_{\sigma_i j} = A_i \omega_j\}.$$

The solenoid is a compact subgroup of $(T^m)^{\mathbb{Z}^k}$ with the product topology. Its dual is a subgroup of $\mathbb{Q}^m$, more precisely it is contained in $(\mathbb{Z}(p_1, \ldots, p_l))^m$ where $p_1, \ldots, p_l$ are those prime integers which occur in the prime decomposition of the determinant of at least one of the matrices $A_1, \ldots A_k$ and $\mathbb{Z}(p_1, \ldots, p_l)$ is the subgroup of rational numbers whose denominators are only divisible by $p_1, \ldots, p_l$. Note that $\mathbb{Z}^k$ acts on S naturally by coordinate shifts. Let us denote this action by α^*. The solenoid is a fibration over T^m with Cantor set fibers by mapping $\omega \in S$ to $\omega(0, \ldots, 0)$. The projection intertwines the α^*-action restricted to $\mathbb{Z}_+^k$ with α. Note that a local cross-section to this fibration is given by the local connected component of ω. We will call this transversal the *toral direction* at ω. Note that the projection is one-to-one if and only if all A_i are invertible.

Every α-invariant measure lifts in a unique fashion to an α^*-invariant measure on S.

There is a natural Hölder structure on the solenoid which comes from any metric on the product of the form

$$d_c(\omega, \omega') = \sum_j \frac{d_{T^m}(\omega_j, \omega'_j)}{c^{\|j\|}}$$

where where $c > 1$ and d_{T^m} is the standard metric on the torus. Note that the Hölder structure is independent of c. This also allows us to define exponential convergence along the fiber and hence stable, unstable and neutral spaces for the elements of α^*.

For most of the paper, and in particular for the Main Theorem 5.1 , it is sufficient to have these rough dynamical structures. For certain applications in Section 6 however, we need to define specific exponential speeds of expansion and contraction, in the p-adic directions i.e. Lyapunov exponents. To do this, we need a more subtle metric structure on S which requires an alternative, more arithmetic description of the solenoid. Its main advantage is that we can canonically define a special metric d on S which gives a Lipschitz structure on S and defines Lyapunov exponents on S which agree with the standard Lyapunov exponents in the toral direction. The metric d is Hölder equivalent to d_c. Since these issues are irrelevant to the invertible case and the Main Theorem, we only give this description in an appendix.

4 Conditional measures and entropy

Let us briefly recall how a probability measure ν on M determines a system of conditional measures on a foliation $\boldsymbol{F}$. Denote by $\mathcal{B}$ the Borel σ-algebra on M. A *measurable partition* ξ of M is a partition of M such that, up to a set of measure 0, the quotient space M/ξ is separated by a countable number of measurable sets [25]. For every x in a set of full ν-measure there is a probability measure ν_x^ξ defined on $\xi(x)$, the element of ξ containing x, and satisfying the following properties: If $\mathcal{B}_\xi$ is the sub-σ-algebra of $\mathcal{B}$ whose elements are unions of elements of ξ, and $A \subset M$ is a measurable set, then $x \mapsto \nu_x^\xi(A)$ is $\mathcal{B}_\xi$-measurable and $\nu(A) = \int \nu_x^\xi(A)\nu(dx)$. These conditions determine the measures ν_x^ξ uniquely.

Given a continuous foliation $\boldsymbol{F}$, let $\boldsymbol{F}(x)$ denote the leaf through x. The partition into the leaves of $\boldsymbol{F}$ is not a measurable partition in general. (Although the point of the Proposition below as well as of the most of the arguments in Section 5 is that in the zero entropy situation they in fact are measurable.) Let $\sigma(\boldsymbol{F})$ denote the σ-algebra of all sets that consist a.e.. of complete leaves of $\boldsymbol{F}$. It corresponds to a unique measurable partition which is called the *measurable hull* of $\boldsymbol{F}$, and is denoted by $\xi(\boldsymbol{F})$. It is the finest measurable partition whose elements consist a.e. from the entire leaves of $\boldsymbol{F}$. Unless it is trivial, it is usually hard to describe geometrically. We will be primarily interested in the intergal foliations of various distributions described in Section 2. Conditional measures on leaves of such a foliation are σ-finite locally finite measures $\nu_x^{\boldsymbol{F}}$ defined up to a multiplicative constant. In other words, for almost every $x \in M$ and for open sets $A, B \subset \boldsymbol{F}(x)$ with compact closures one can canonically define *the ratio* $\frac{\nu_x^{\boldsymbol{F}}(A)}{\nu_x^{\boldsymbol{F}}(B)}$.

In the homogenous case in question as well as in some other cases this can be done as follows. Take a small homogenous transversal T to $\boldsymbol{F}(x)$ at x and translate it to cover a neighborhood of large enough disc D in $\boldsymbol{F}(x)$ which contains both A and B. Thus in this neighborhood we have a product structure modeled on $D \times T$. There is also a metric which is translation invariant. Let $T(\epsilon) \subset T$ be the ϵ ball around x. Then

$$\frac{\nu_x^{\boldsymbol{F}}(A)}{\nu_x^{\boldsymbol{F}}(B)} = \lim_{\epsilon \to 0} \frac{\nu(A \times T(\epsilon))}{\nu(B \times T(\epsilon))}.$$

There is an alternative way of describing conditional measures which works in a more general situation. Call a measurable partition ξ *subordinate* to $\boldsymbol{F}$ if for ν-a.e. x we have $\xi(x) \subset \boldsymbol{F}(x)$ and $\xi(x)$ contains a neighborhood of x open in the submanifold topology of $\boldsymbol{F}(x)$. Note that two different partitions subordinate to the same foliation determine conditional measures that are scalar multiples when restricted to the intersection of an element of one partition with an element of the other partition. Thus there is a locally finite measure $\nu_x^{\boldsymbol{F}}$ on $\boldsymbol{F}(x)$ uniquely defined up to scaling that restricts to a scalar multiple of a conditional measure for each partition subordinate to $\boldsymbol{F}$. The measures $\nu_x^{\boldsymbol{F}}$ form the system of conditional measures on the leaves of $\boldsymbol{F}$. In more general situations which do not concern us in this paper a certain care is needed to justify the fact that conditional measures are really correctly defined up to a constant scalar multiple. However in order to show connections between trivialization of conditional measures it is enough to see that conditional measures are defined up to a scalar function which is of course quite obvious from the preceeding construction in a fairly great generality. Of course, at the end this is not surprising at all since the conclusion will be that the conditional measures are atomic, hence finite and can be normalised so that the partition into leaves is measurable!

Given $a \in \mathbb{R}^k$ and an a-invariant measure μ, we denote the partition into ergodic components of μ under a by ξ_a.

Let us recall the relation between conditional measures and entropy. It is well-known that entropy is related to exponential contraction and expansion. In order to accomodate solenoids, we will formulate a criterion for the vanishing of metric entropy in the context of foliated compact metric spaces.

The underlying spaces for our actions are locally isometric with the product of some $\mathbb{R}^n$ with finitely many $\mathbb{Q}_p^{m_p}$. All the invariant distributions and associated foliations are also locally isometric to such products. Recall that the *box dimension* of a metric space (M, d) is given by

$$\limsup_{\varepsilon \to 0} -\frac{\log(N_d(\varepsilon))}{\log(\varepsilon)}$$

where $N_d(\varepsilon)$ is the maximum number of disjoint ε-balls in M. Let $\varepsilon > \delta > 0$ and let $N_d(\varepsilon, \delta)$ be the maximum number of disjoint δ-balls in any ε-ball. there is a constant $D > 0$ such that for all small $\varepsilon > \delta > 0$,

$$\frac{\log N_d(\varepsilon,\delta)}{\log(\frac{\varepsilon}{\delta})} < D.$$

Let (F, d_F) and (T, d_T) be metric spaces. A *foliation* $\boldsymbol{F}$ of a metric space (M, d), modeled on F with transversal T is a disjoint decomposition of M into subspaces F_x, called the *leaves* of $\boldsymbol{F}$ such that each F_x is the Lipschitz image of F and for every point $x \in X$, there is a neighborhood U such that U is bi-Lipschitz with the metric product $U_F \times U_T$ where U_f and U_T are neighborhoods in F and T respectively, and where the bi-Lipschitz map takes $U_F \times \{t\}$ for all $t \in U_T$ to the intersection of a leaf of $\boldsymbol{F}$ with U. We say that a pair of foliations $\boldsymbol{F}$ and $\boldsymbol{G}$ of M define a *local product structure* if $\boldsymbol{F}$ is modelled on F with transversal T and $\boldsymbol{G}$ is modelled on T with transversal F and the bi-Lipschitz maps defined locally respect both foliations simultaneously.

Proposition 4.1 *Let M be a compact metric space of finite local box dimension. Suppose $\boldsymbol{F}$ and $\boldsymbol{G}$ are foliations on M that define a local product structure. Let $\phi : M \to M$ be a bi-Lipschitz homeomorphism preserving $\boldsymbol{F}$ and $\boldsymbol{G}$, which locally strictly contracts $\boldsymbol{F}$ and such that for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all $n \geq 0$ and $y \in \boldsymbol{G}(x)$ the distance $d(\phi^{-n}(y), \phi^{-n}(x)) \leq C_\varepsilon e^{\varepsilon n}\, d(x, y)$ if $d(x, y) < \varepsilon$.*

Then, if μ is a Borel probability measure on M, and $\mu_x^{\boldsymbol{F}}$ its system of conditional measures, the metric entropy $h_\mu = 0$ if and only if for μ-a.e. x, $\mu_x^{\boldsymbol{F}}$ is atomic. In this case, $\mu_x^{\boldsymbol{F}}$ is supported on a single point.

In this paper, we will only need this statement in one direction, namely that the metric entropy is 0 if the conditional measures $\mu_x^{\boldsymbol{F}}$ are atomic. We will describe the proof of this direction. First, note that the conditional measure $\mu^{\boldsymbol{F}}$ is supported on a single point if it is atomic. Indeed, if x is an atom of the conditional measure, there is a small neighborhood U of x in the leaf such that $\mu_x^{\boldsymbol{F}}(U - \{x\}) < \varepsilon\, \mu_x^{\boldsymbol{F}}(\{x\})$. Pushing $\mu_x^{\boldsymbol{F}}$ backward and using Poincaré recurrence, we see that for a typical x, $\mu_x^{\boldsymbol{F}}$ is concentrated at x.

Now assume that $\mu_x^{\boldsymbol{F}}$ is supported in a single point. Then we can find a set of full measure which intersects every $\boldsymbol{F}$-leaf in at most one point. In particular, the intersection of this set with a neighborhood with local product structure is the graph of a measurable function defined on an open set $U \subset T$ with values in an open set $V \subset F$. By Lusin's theorem, there is a compact set K of arbitrarily large measure which is a finite union of graphs of continuous maps from subsets of T to F. Let L be a Lipschitz constant for ϕ. Pick an n and $\delta > 0$ such that $L^n\,\delta$ is small. Consider a partition ξ of M with two types of elements: intersections of sets of diameter less than δ with K and with $M \setminus K$. It is well known that $h(\phi, \xi) \leq \frac{1}{n} H(\xi, \phi^{-n}\xi)$. The latter quantity is estimated separately for the preimages under ϕ^{-n} of the two types of elements of ξ. In both cases, we just estimate the contribution of each element $c \in \phi^{-n}\xi$ by the number of elements of ξ which have non-empty intersections with c. For c in $\phi^{-n}K$, we estimate the diameter of c by $C_\varepsilon\, e^{\varepsilon\, n}\,\delta$, using the assumption of the proposition. Since the local box dimension is finite, the number of non-trivial intersections grows at most exponentially in n with arbitrarily small exponent.

For $\phi^{-n}(c)$ for $c \in \xi$ where $c \subset M \setminus K$, we have a uniform exponential estimate of the size and hence number of nontrivial intersections using the Lipschitz constant of ϕ. Since the measure of $M \setminus K$ is small, the contribution of such elements to the conditional entropy is small.

5 The main theorem for toral endomorphisms

In all standard toral examples, let us consider the tangent bundle to the phase space and define the derivative action. (In the solenoid case, this will include some non-Archimedean components, as explained in the Appendix. All the arguments in this section however only use the Archimedean directions. Thus we may just consider the real tangent bundle over the solenoid in this section.) The derivative is a linear extension of the action, and the Lyapunov exponents are given by logarithms of the appropriate valuations of the eigenvalues. By commutativity we can find a joint splitting into subspaces on which the Lyapunov exponents are constant for each element. This is the decomposition into Lyapunov spaces described in Section 2.

Let us recall again that there is a one-to-one correspondence between Borel probability ergodic invariant measures for an action of $\mathbb{Z}_+^k$ by toral endomorphisms and those for the $\mathbb{R}^k$-action which is the suspension of the solenoid extension of the toral action. Since in our arguments we will be dealing mostly with $\mathbb{R}^k$-actions obtained as suspensions of solenoid extensions we will adopt the following notation. If μ is an invariant measure for an $\mathbb{R}^k$-action then μ_{T^m} will denote the corresponding measure for the toral action. Obviously, every element of the $\mathbb{R}^k$-action has zero entropy w.r.t. μ if and only if every element of the corresponding $\mathbb{Z}_+^k$-action has zero entropy w.r.t. μ_{T^m}.

The following theorem is our principal technical result in the toral case.

Theorem 5.1 *Let α be a $\mathbb{R}^k$-action with $k \geq 2$ induced from a standard action by toral endomorphisms. Assume that μ is an ergodic invariant measure for α such that there are generic singular elements $a_1, \ldots, a_k$ and a regular element $b \in \mathbb{R}^k$ with E_b^+ totally Archimedean such that*

$$(*) \qquad E_b^+ = \sum_i (E_{a_i}^0 \cap E_b^+)$$

(where the sum need not be direct) and such that

$$(**) \qquad \xi_{a_i} \leq \xi(E_{a_i}^0 \cap E_b^+).$$

Then either μ_{T^m} is Haar measure on a rational subtorus or every element of α has 0 entropy w.r.t. μ.

The genericity of the a_i is not actually needed as one can easily see from the proof of the theorem.

Remark : This theorem generalizes to suspensions of groups of solenoid automorphisms more general than those obtained from extensions of groups of toral endomorphisms (cf. Example 3.6). The principal difference in the formulation is that the stable distribution E_b^+ is not assumed to be totally Archimedean. Conditions (*) and (**) remain the same. The differences in the proof are not very significant, and will be left to the reader.

Corollary 5.2 *Let α be a standard $\mathbb{R}^k$-action with $k \geq 2$ induced from an action by toral endomorphisms. Let μ be an α-invariant measure such that every one-parameter subgroup is ergodic. Then either μ_{T^m} is Haar measure on a rational subtorus or every element of α has 0 entropy w.r.t. μ.*

Since mixing for an action of $\mathbb{Z}_+^k$ implies that every one-parameter subgroup of the suspension of its natural extension is ergodic, we have the following corollary.

Corollary 5.3 *Let α be a standard toral $\mathbb{Z}_+^k$ or $\mathbb{Z}^k$ action with $k \geq 2$. Then every mixing α-invariant measure ν is either Haar measure on a rational subtorus or every element has 0 entropy w.r.t. ν. In particular, if there are no rational invariant subspaces, then ν is Haar measure on the torus itself or every element has 0 entropy w.r.t. ν.*

Proof of Theorem 5.1: First let us describe the scheme of the proof. Let b be as in the statement of the theorem. We may assume without loss of generality that different Lyapunov exponents take different values at b by moving b within its Weyl chamber if necessary.

We will show that the conditional measure of the stable foliation of b is atomic unless μ_{T^m} is Haar measure on a rational subtorus. By Proposition 4.1, the entropy of b is 0. Since the stable foliation of any two elements in the same Weyl chamber coincide, this also implies that the entropy of any element in the Weyl chamber $\mathcal{C}$ of b is zero. Since entropy is invariant under taking inverses, all entropies of elements in $-\mathcal{C}$ vanish. Now every element can be represented as a positive linear combination of elements in $\mathcal{C}$ and $-\mathcal{C}$. Since the entropy is sublinear [7], we see that all entropies are zero.

As noted in Section 2, the center foliation of a singular element in $\mathbb{R}^k$ always contains a non-trivial isometric direction. This follows from the Jordan decomposition. Furthermore, the $\boldsymbol{p}$-adic parts of the center direction are all isometric.

We first show that the conditional measure of μ w.r.t. $W_b^+ \cap W_{a_i}^I$ is either atomic or μ is Haar measure on a rational invariant subtorus. This is done by Lemmata 5.4 - 5.8. In Lemma 5.4 we analyze the invariance properties of the conditional measure w.r.t. $W_b^+ \cap W_{a_i}^I$ along fibers of $W_b^+ \cap W_{a_i}^I$. This is the only place where the condition (**) is used. In Lemma 5.5, we restrict our considerations to the intersection of $W_{a_i}^I$ with a single Lyapunov subspace and deduce that the support of the corresponding conditional measure is an affine subspace. Using Lemma 5.4, we establish in Lemma 5.6 that this conditional measure is Haar on this subspace. If this subspace has positive dimension then μ is invariant under a one-parameter group of translations. Unique ergodicity of such a group on its orbit closure implies that μ_{T^m} is Haar measure on an invariant rational subtorus (cf. Lemma 5.7). Then we proceed by induction on the Lyapunov exponents and establish the desired dichotomy.

Next, in Lemma 5.9, we establish the same dichotomy for $W_b^+ \cap W_{a_i}^0$. Thus we can assume for the rest of the proof that for all i, the conditional measures of μ on any $W_b^+ \cap W_{a_i}^0$ are atomic.

We then conclude that the conditional measure w.r.t. stable foliation of b is supported on the invariant complement of $W_{a_i}^0 \cap W_b^+$ in W_b^+ (cf. Lemma 5.10). This is the beginning of another induction. For that purpose, we restrict α to a 2-plane $\mathcal{P}$ which contains b and intersects all the Lyapunov hyperplanes in generic lines. Pick $c_i \neq 0$ in the intersection of $\mathcal{P}$ with the unique Lyapunov hyperplane that contains a_i. Since any two generic elements in the same Lyapunov hyperplane have the same center foliations, we can replace the $a_i's$ by the $c_i's$ in condition (*). Hence the conditional measures of μ on any $W_b^+ \cap W_{c_i}^0$ are atomic. Reorder the c_i such that the indexing of the c_i's corresponds with the ordering of the Lyapunov lines on $\mathcal{P}$ starting from $\mathcal{C}$. Note that not every Lyapunov line necessarily contains a c_i. In fact, those that do contain a c_i correspond to the kernels of Lyapunov exponent negative on b. Denote the invariant complement of $W_{c_1}^0 \cap W_b^+$ in W_b^+ by $\boldsymbol{G}_1(x)$. Then $\boldsymbol{G}_1(x)$ can be split into a component in the center manifold of c_2 and a sum $\boldsymbol{G}_2(x)$ of Lyapunov spaces. As for c_1, we show that the conditional measure w.r.t. the stable foliation of b is supported on a single leaf of $\boldsymbol{G}_2(x)$ (cf. Lemma 5.10). Continuing in this fashion, using Lemma 5.10 inductively, we see that the support of the conditional measure w.r.t. stable foliation of b is contained in smaller and smaller leaves of foliations, and eventually will shrink to a point. This follows from the condition (*). Hence b has 0 entropy.

Now we proceed to the details. We will call a foliation *Archimedean* if the leaves do not contain any non-Archimedean directions. Suppose that a is a generic singular element. Let $\boldsymbol{F}(x) \subset W_a^I$ be any a-invariant Archimedean subfoliation of W_a^I. Denote by $B_1^{\boldsymbol{F}}(x)$ the unit ball in $\boldsymbol{F}(x)$ about x with respect to the flat metric. Let $\mu_x^{\boldsymbol{F}}$ denote the system of conditional measures determined by $\boldsymbol{F}$ normalized by the requirement that $\mu_x^{\boldsymbol{F}}(B_1^{\boldsymbol{F}}(x)) = 1$ for all x in the support of μ.

The next lemma contains the key geometric idea of the proof. We use the fact that singular elements are isometric in certain directions together with condition (**) to show invariance of the system of conditional measures under a certain group of isometries.

Lemma 5.4 *Suppose that $\xi_a \leq \xi(\boldsymbol{F})$. Then for μ-a.e. x, the support of $\mu_x^{\boldsymbol{F}}$ is the orbit of the closed subgroup G_x of isometries of $\boldsymbol{F}(x)$ which preserve $\mu_x^{\boldsymbol{F}}$ up to a scalar multiple. Furthermore, for $\mu_x^{\boldsymbol{F}}$-a.e. $y \in \boldsymbol{F}(x)$, $\mu_y^{\boldsymbol{F}}$ is the image of $\mu_x^{\boldsymbol{F}}$ under an isometry in G_x.*

Proof: Since G_x maps the support of $\mu_x^{\boldsymbol{F}}$ to itself we only need to show that G_x is transitive on the support of $\mu_x^{\boldsymbol{F}}$.

Let K_0 be the set of all x such that the ergodic component of a passing through x contains $\boldsymbol{F}(x)$ (up to a set of $\mu_x^{\boldsymbol{F}}$-measure 0). By the assumption on a, K_0 has full μ-measure.

For μ-a.e. x, the ergodic component $\boldsymbol{E}_x$ of x is well-defined. Let μ_x be the induced measure on $\boldsymbol{E}_x$. By Lusin's theorem, for every $\varepsilon > 0$ there is a closed set K_1 contained in the support of μ such that

1. $\mu_x(\boldsymbol{E}_x \cap K_1) > 1 - \varepsilon$ for all $x \in K_1$, and
2. $\mu_x^{\boldsymbol{F}}$ depends continuously on $x \in K_1$ w.r.t. the weak*-topology.

Set $K_2 = K_0 \cap K_1$. Since the transformation induced by a on $K_2 \cap \boldsymbol{E}_x$ is ergodic, the set K_3 of $x \in K_2$ whose orbit $\{a^n x\}_{n \in \mathbb{Z}}$ is dense in $K_2 \cap \boldsymbol{E}_x$ has full μ-measure in K_2. Finally, for any constant $L > 0$, let K_4 be the set of all $x \in K_3$ such that $\mu_x^{\boldsymbol{F}}(\boldsymbol{F}(x) \cap K_3 \cap B_1^{\boldsymbol{F}}(x)) > 1 - L\,\varepsilon$, where again, $B_1^{\boldsymbol{F}}(x)$ is the ball of radius 1 in $\boldsymbol{F}(x)$ w.r.t. the flat metric. Then $\mu(K_4) > (1 - \frac{1}{L})\mu(K_3) > (1 - \frac{1}{L})(1 - \varepsilon)$. If $x \in K_4$ and $y \in \boldsymbol{F}(x) \cap K_3$, there is a sequence

$n_k \to \infty$ with $a^{n_k} x \in K_4$ converging to y. Since $\mu_x^{\boldsymbol{F}}$ depends continuously on $x \in K_4$ w.r.t. the weak*-topology and a maps $\boldsymbol{F}(x)$ isometrically, it follows that $\mu_y^{\boldsymbol{F}} = \phi \, \mu_x^{\boldsymbol{F}}$ where ϕ is some isometry of $\boldsymbol{F}(x)$. Since $\mu_y^{\boldsymbol{F}}$ is a scalar multiple of $\mu_x^{\boldsymbol{F}}$, ϕ belongs to G_x. Since ε can be chosen arbitrarily small and L arbitrarily large we see that there is a set X of μ-measure 1 such that for any $x \in X$ and y in the support of $\mu_x^{\boldsymbol{F}}$, y is in the G_x-orbit of x. Since the support of $\mu_x^{\boldsymbol{F}}$ is closed and the orbit of G_x of x is closed, the claim follows. ◇

Lemma 5.5 *In addition to the assumptions in Lemma 5.4, let $\boldsymbol{F}$ be contained in the intersection of W_a^I with a Lyapunov subspace for a non-zero Lyapunov exponent λ. Then for μ-a.e. x, the support S_x of $\mu_x^{\boldsymbol{F}}$ is an affine subspace of $\boldsymbol{F}(x)$.*

Proof: By the last lemma, S_x is the orbit of a closed group of isometries. Therefore S_x is a submanifold, possibly disconnected. Note that the maximal principal curvature of S_x is constant along S_x. Let $\kappa(x)$ denote this constant.

Let c be any element such that $\lambda(c) < 0$. Note that c maps S_x to S_{cx}. Iterates of c exponentially contract the fibers of $\boldsymbol{F}$. In particular, since the exponential contractions in all directions inside $\boldsymbol{F}$ are the same, any curve with positive principal curvature will be mapped to curves with exponentially increasing principal curvatures. Hence $\kappa(c^n x)$ goes to infinity for μ-a.e. x unless $\kappa(x) = 0$. This is impossible by Poincaré recurrence. Thus $\kappa(x) \equiv 0$, and hence the support of $\mu_x^{\boldsymbol{F}}$ is a union of non-intersecting affine subspaces.

Let us now show that the support is connected. Suppose to the contrary that the support is a union $\cup A_i$ of at least two affine subspaces A_i. Let d_x denote the minimum of the distances from x to any A_i which does not contain x. Since the support is a closed subset, $d_x > 0$ for all x. Note that $d_{c^n x} \to 0$ as $n \to \infty$. This is again a contradiction to Poincaré recurrence. ◇

Lemma 5.6 *Under the assumptions of Lemma 5.5, $\mu_x^{\boldsymbol{F}}$ is Haar measure on S_x.*

Proof: By Lemma 5.4 the group G_x of isometries of $\boldsymbol{F}(x)$ which map $\mu_x^{\boldsymbol{F}}$ to a scalar multiple acts transitively on the support S_x of $\mu_x^{\boldsymbol{F}}$. By Lemma 5.5, S_x is an affine space. Let $G_x = K_x T_x$ be the decomposition of G_x into a product of the solvable radical subgroup T_x by a semisimple group K_x, the so-called Levi component. Note that T_x is abelian and uniquely defined as T_x is normal. Since G_x is transitive on the affine space S_x, so is T_x. Since $T_x \subset G_x$ we can define a measurable cocycle $c : T_x \times S_x \to \mathbb{R}_+$ by the relation

$$c(g, y)\, \mu_y^{\boldsymbol{F}} = \mu_{g\,y}^{\boldsymbol{F}}.$$

If $h \in G_x$, note that for $y \in \boldsymbol{F}(x)$ the pushforward $h_* \mu_y^{\boldsymbol{F}} = \mu_{h\,y}^{\boldsymbol{F}}$ since h maps $\mu_y^{\boldsymbol{F}}$ to a scalar multiple by definition and preserves the normalization as h is an isometry. Since T_x is abelian and $c(g,x)\mu_x^{\boldsymbol{F}} = \mu_{g\,x}^{\boldsymbol{F}}$, we find for all $g, h \in T_x$ that

$$c(g,x)\mu_{hx}^{\boldsymbol{F}} = c(g,x) h_* \mu_x^{\boldsymbol{F}} = h_* \, \mu_{g\,x}^{\boldsymbol{F}} = \mu_{h\,g\,x}^{\boldsymbol{F}} = \mu_{g\,h\,x}^{\boldsymbol{F}} = c(g, h\,x)\mu_{hx}^{\boldsymbol{F}}.$$

This shows that c is $\mu_x^{\boldsymbol{F}}$-a.e. constant in S_x, and hence defines measurable homomorphisms $c_x : T_x \to \mathbb{R}_+$.

Since b maps conditional measures to conditional measures, b induces a homomorphism $G_x \to G_{bx}$ via

$$g \mapsto (b\,|_{\boldsymbol{F}(x)}) \circ g \circ (b^{-1}\,|_{\boldsymbol{F}(bx)}).$$

Denote the induced homomorphism on the solvradicals by $b_x : T_x \to T_{bx}$. Note that

$$(*) \qquad c_{bx} \circ b_x = c_x.$$

Indeed, we have

$$c_{bx}(b_x(g))\mu^{\boldsymbol{F}}_{bx} = \mu^{\boldsymbol{F}}_{b_x(g)\,bx} = \mu^{\boldsymbol{F}}_{b \circ g \circ b^{-1}(bx)} = \mu^{\boldsymbol{F}}_{b \circ g(x)} = b_* \mu^{\boldsymbol{F}}_{g(x)} = b_*(c_x(g)\mu^{\boldsymbol{F}}_x) = c_x(g)\mu^{\boldsymbol{F}}_{bx}.$$

By assumption, $\boldsymbol{F}$ is contained in the λ-Lyapunov space, and $\lambda(b) \neq 0$. As T_y acts simply transitively on $\boldsymbol{F}(y)$, we can endow the T_y with the induced Riemannian metric $\| \, \|_y$ from the torus. Also we can identify T_y with a subgroup of the torus. Let K be a compact set of positive measure on which T_y, $\| \, \|_y$ and c_y vary continuously. Using recurrence on K once again, we see that the homomorphisms $(b^n)_x : T_x \to T_{b^n x}$ expand exponentially fast at least on recurrent subsequences in K. By (*), this contradicts the continuity of c_y on K. ◇

Lemma 5.7 *Let $\boldsymbol{G}$ be a linear foliation on T^m. Suppose that ν is a probability measure on T^m such that for ν-a.e. $x \in T^m$, the conditional measure $\nu_x^{\boldsymbol{G}}$ on $\boldsymbol{G}(x)$ is Lebesgue. Then ν is Lebesgue on a subtorus of T^m saturated by $\boldsymbol{G}$.*

Proof: The condition implies that ν is invariant under the subgroup G of T^m whose orbit foliation is $\boldsymbol{G}$. Hence ν is invariant under the closure of G in T^m, and hence Haar on the corresponding subtorus. ◇

Recall that b is a regular element satisfying (*). For the remainder of the proof, we assume that a is a generic singular element satisfying condition (**).

Lemma 5.8 *Let $\boldsymbol{F}$ be the foliation $W_a^I \cap W_b^+$. For μ-a.e. x, the conditional measure $\mu_x^{\boldsymbol{F}}$ is atomic unless μ_{T^m} is Haar measure on a subtorus.*

Proof: Split $\boldsymbol{F} = \sum_\lambda (\boldsymbol{F} \cap W^\lambda)$ into its intersection with the Lyapunov subspaces. By Lemmata 5.4, 5.6 and 5.7 the support S_x of $\mu_x^{\boldsymbol{F}}$ is a smooth submanifold which intersects every $\boldsymbol{F} \cap W^\lambda$ in at most one point unless μ is Haar measure on a subtorus. Let λ be the Lyapunov exponent smallest on b. Let D be the distribution of tangent spaces of S_x. It is measurable, and b-invariant and C^∞ on $\boldsymbol{F}(x)$. Since D cannot intersect the component in the E_λ-direction in a subspace of positive dimension, D must lie in the sum $\sum_{\mu \neq \lambda} E_\mu$ by b-invariance. By induction, by taking the Lyapunov exponents in increasing order, we see that D is trivial and $\mu_x^{\boldsymbol{F}}$ atomic unless μ is Haar on a subtorus. ◇

Let a be a generic element on the Lyapunov hyperplane defined by a Lyapunov exponent λ for which $\lambda(b) < 0$. Denote by W^0 the foliation with tangent distribution $E^0 = E_a^0 \cap E_b^+$. Let μ_x^0 denote the system of conditional measures with respect to $W^0(x)$ normalized by $\mu_x^0(B_1^0(x)) = 1$ where $B_1^0(x)$ is the unit ball in $W^0(x)$ about x with respect to the flat metric.

The argument for the next lemma is another application of Lusin's theorem. The main construction will be used again in the next lemma.

Lemma 5.9 *Let a be as above. Suppose that the conditional measures on $\boldsymbol{F}_I = W_a^I \cap W_b^+$ are atomic. Then the conditional measure μ^0 on $\boldsymbol{F} = W_a^0 \cap W_b^+$ is atomic.*

Proof: By the Jordan decomposition, $\boldsymbol{F}_I$ is non-trivial if $\boldsymbol{F}$ is non-trivial.

By ergodicity of μ, the conditional measure μ_x^0 is either continuous for a.e. x or atomic for a.e. x.

Let $K \subset M$ be compact such that for all $x \in K$, $K \cap \boldsymbol{F}_I(x)$ is at most one point and $\mu(K) > 1 - \frac{1}{L}$ for some L to be determined later. Such a K can be found by approximating from inside a set of full measure which intersects any fiber of $\boldsymbol{F}_I$ in at most one point. Since μ_x^0 is continuous we can further assume that for every $x \in K$, $K \cap W^0(x)$ has no isolated points in the topology of the subspace.

Let $D(x)$ be the orthogonal complement to $\boldsymbol{F}_I(x)$ in $W^0(x)$. Denote by $B_\alpha^D(x)$ (respectively $B_\alpha^{\boldsymbol{F}_I}(x)$) the α-ball about x in $D(x)$ (respectively $\boldsymbol{F}_I(x)$). One can choose $\varepsilon > 0$ and $\delta > 0$ such that for every $x \in K$, $K \cap (B^{\boldsymbol{F}_I}(x) \times B_\delta^D(x))$ is the graph of a function $\phi_x : P_x \to B_\varepsilon^{\boldsymbol{F}_I}(x)$ and ϕ_x is an equicontinuous family of functions (with domains P_x varying continuously in the Hausdorff topology).

We will arrive at a contradiction by showing that if $x \in K$ returns to K under the action of an appropriately chosen element of $\mathbb{R}^k$ the pushforward of ϕ_x will become too steep, and hence cannot belong to an equicontinuous family.

By compactness and perfectness of K, there are numbers $\delta > \delta_1 > 0$ such that for all $x \in K$, $K \cap (B_\varepsilon^{\boldsymbol{F}_I}(x) \times (B_\delta^D(x) \setminus B_{\delta_1}^D(x))) \neq \emptyset$. By equicontinuity there is $\delta_2 > 0$ and such that for $x \in K$ and $y \in B_{\delta_2}^D$ we have $\phi_x(y) \in B_{\varepsilon/100}^{\boldsymbol{F}_I}(x)$.

Let $p \in B_\varepsilon^{\boldsymbol{F}_I}(x) \times (B_\delta^D(x) \setminus B_{\delta_1}^D(x))$ for $x \in K$. Combining the finitely many exponential speeds of decay for b with the finitely many polynomial speeds of expansion for a, we can pick a finite number of elements $t_i b + n_i a \in \mathbb{R}^k$ for $i = 1, \ldots, L$ such that $(t_i b + n_i a)(p) \subset (\boldsymbol{F}_I((t_i b + n_i a)\, x) \setminus B_{\varepsilon/100}^{\boldsymbol{F}_I}((t_i b + n_i a)\, x)) \times B_{\delta_2}^D((t_i b + n_i a)\, x)$ for at least one i. Let us point out that the number L depends only on a and b and hence can be chosen in advance while the specific numbers t_i and n_i depend on δ_2 and ε and hence have to be determined once K is fixed.

Since $\mu(K) > 1 - \frac{1}{L}$ we can find $x \in K$ such that $(t_i b + n_i a)\, x \in K$ for all i. By the above there is a $p \in B_\varepsilon^{\boldsymbol{F}_I}(x) \times (B_\delta^D(x) \setminus B_{\delta_1}^D(x))$. Then for some i, $(t_i b + n_i a)\,(p) \in (\boldsymbol{F}_I((t_i b + n_i a)\, x) \setminus B_{\varepsilon/100}^{\boldsymbol{F}_I}((t_i b + n_i a)\, x)) \times B_{\delta_2}^D((t_i b + n_i a)\, x)$ in contradiction to the choice of δ_2. ◇

The proof of the next lemma is very similar to that of the last lemma.

Lemma 5.10 *Suppose that A, B and H are invariant subfoliations of W_b^+, $H = A \oplus B$, A belongs to W_a^0 and B belongs to W_a^+. Then the conditional measure of μ on H is supported on a single leaf of B.*

Proof: By Lemma 5.9, there is a set of full measure which intersects almost every leaf of $W_a^0 \cap W_b^+$ and hence A in at most one point. Hence for μ-a.e. x, the intersection with almost every leaf of H has locally the form of a graph of a measurable function $\phi_x : B(x) \to A(x)$. Since b is ergodic and contracts, it follows that either for μ-a.e. x, $\phi_x = 0$ or $\phi_x \neq 0$. As in the proof of the previous lemma, we can pick a compact set K such that for all $x \in K$, $K \cap A(x)$ is at most one point and $\mu(K) > 1 - \frac{1}{L}$ where L is as in the previous lemma. Furthermore, there is $\varepsilon > 0$ as before and $\delta > 0$ such that $\| \phi_x(y) \| > \delta$ for some $y \in B_\varepsilon^b(x)$. As before, $K \cap H(x)$ is the graph of a continuous function ϕ_x defined on a closed set which can be assumed to be perfect. Moreover, the ϕ_x vary continuously.

We will now see that the set $K \cap H(x)$ is contained in the tubular neighborhood of $B(x)$ of radius δ, similarly to the proof of Lemma 5.9. Indeed, let y be outside the δ-tubular neighborhood. By equicontinuity of the family of functions ϕ_x one can choose such a y with the B-coordinate which is small enough but away from zero. Then there are finitely many sequences in the plane generated by b and a such that at least one of them moves every such y to $y' \neq x$ in $A(x)$. Then picking a sufficiently large element in each of these sequences, we deduce that the set $K \cap H(x)$ is contained in the tubular neighborhood of $B(x)$ of radius δ. Since δ can be chosen arbitrary small the statement of the lemma follows. $\diamond$

Now Theorem 5.1 follows as described in the outline of the proof.

6 Rigidity of positive entropy ergodic measures for toral endomorphisms

In a number of cases, the technical condition (**) on the measure can be deduced from just ergodicity of the measure with respect to the action of the whole (semi) group. The key step for this is the following lemma. Denote the measurable hull $\sigma(W_a^+)$ of the unstable foliation of an element $a \in \mathbb{R}^k$ by $\mathcal{W}_a^+$. Similarly let $\mathcal{W}_a^- = \sigma(W_a^-)$ and $\mathcal{W}_a^0 = \sigma(W_a^0)$ be the measurable hulls of the stable and neutral foliations of a respectively.

Lemma 6.1 *Suppose $a, b \in \mathbb{R}^k$ such that b is Anosov and $E_b^+ \subset E_a^+$. Then we have $\xi_a \leq \mathcal{W}_a^0$.*

Proof: The Hopf argument shows that $\xi_a \leq \mathcal{W}_a^+$. Indeed, if f is a continuous function on M then the forward ergodic averages $F^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(a^n\, x)$ are constant along stable manifolds of a as they contract exponentially under a. Since the continuous functions are dense in $L_\mu^2(M)$, it follows that any invariant L_μ^2-function is constant a.e. on $W_a^+(x)$ with respect to the conditional measure induced by μ.

Since $E_b^+ \subset E_a^+$ we get $\mathcal{W}_a^+ \leq \mathcal{W}_b^+$. Note that $\mathcal{W}_b^- = \mathcal{W}_b^+$ since they both equal the Pinsker algebra of b. This can be seen as in [1, 20]. Even though some of the directions included in those foliations are non-archimedean the standard argument establishing their equality with the Pinsker algebra works since it only uses exponential expansion and contraction of foliation and does not use the Euclidean structure and works in the framework of Borel measure- preserving homeomorphisms of metric spaces with uniformly contractiong foliations satisfying some mild geometric conditions. More specifically, the argument is based on showing that one can find fine enough partitions ξ with $\mu(\partial\xi) = 0$ such that the intersection of almost every element of the past ξ^- with almost every leaf of the contracting foliation contains an open set in the leaf. Then the negative iterates of ξ^- contain larger and larger balls in the leaves and the infinite past contains the whole leaves. But such partitions can be easily constructed form coverings by closed balls using the following general remark: every ball B can be pertubed to a set B' for which exponentially decreasing neighborhood of the boundary have exponentially decreasing measure.

Now using the decomposition into Lyapunov spaces we see that $E_b^+ \subset E_a^+$ implies that $E_b^- \supset E_a^0 \oplus E_a^- \supset E_a^0$, and hence that $\mathcal{W}_b^- \leq \mathcal{W}_a^0$.

Combining all the inequalities between σ-algebras established above we get

$$\xi_a \leq \mathcal{W}_a^+ \leq \mathcal{W}_b^+ = \mathcal{W}_b^- \leq \mathcal{W}_a^0.$$

$\diamond$

Proposition 6.3 below gives two natural conditions that imply the hypothesis of the lemma above. First we need a simple lemma on Lyapunov exponents. Recall that the Lyapunov exponents of α define linear functionals on $\mathbb{R}^k$.

Lemma 6.2 *Let α be a standard action of $\mathbb{Z}_+^k$, $k \geq 2$. Suppose the action contains an Anosov element a with one-dimensional unstable foliation. Then not all Lyapunov exponents of the suspension of the solenoid extension of α are proportional to each other.*

Proof: Assume to the contrary that all Lyapunov exponents λ_i are proportional. Suppose first that some element in $\mathbb{Z}_+^k$ is not invertible. Then there is at least one non-Archimedean Lyapunov exponent λ' (cf. the Appendix for a discussion of non-Archimedean Lyapunov exponents). Recall that λ' takes rational values on $\mathbb{Z}^k$. Hence the kernel of λ' contains a nontrivial element $b \in \mathbb{Z}^k$. Pick a large l such that $c := a^l b \in \mathbb{Z}_+^k$. Note that c leaves the one-dimensional unstable manifold $W_a^-(0)$ of a through 0 invariant. W.l.o.g. let $\lambda = \lambda_1$ be the Lyapunov exponent determined by the unstable distribution E_a^-. Since λ and λ' are proportional, we see that $\lambda(a^l) = \lambda(c)$. Since $W_a^-(0)$ is one-dimensional, and 0 is fixed, we see that a^l and c act identically on $W_a^-(0)$. Since a is Anosov, $W_a^-(0)$ is dense in T^m. Hence c and a^l define the same endomorphism of T^m. Hence $a^l = c$ as a matrix. Thus we see that $b = 0$, contradicting the choice of b.

Therefore we may suppose that all elements in $\mathbb{Z}_+^k$ are invertible so that we have an action of $\mathbb{Z}^k$ on T^m. Suppose first that all elements are semisimple. By commutativity, there is a matrix M such that for all $c \in \mathbb{Z}^k$, $M^{-1}\,c\,M$ is diagonal (over $\mathbb{C}$). Let $D \geq 1$ be the maximum of the matrix norms $\| M \|, \| M^{-1} \|$ of M and M^{-1}. Then the matrix norm of $c \in \mathbb{Z}^k$ is bounded in terms of its eigenvalues and D and thus by its Lyapunov exponents and D. We get

$$\| c \| \leq D^2 \max_i e^{\lambda_i(c)}.$$

Now we will argue via the covering action of $\mathbb{Z}^k$ on $\mathbb{R}^m$. Let $\boldsymbol{l}$ denote the line through 0 covering $W_a^-(0)$. Pick $v \in \mathbb{Z}^m$, $v \neq 0$ such that

there is an element $v' \in \boldsymbol{l}$ with $\| v - v' \| < 10^{-6} \frac{1}{2D^2}$. Let $\lambda = \lambda_1$ again be the Lyapunov exponent determined by W_a^-. By assumption, there are $c_i \in \mathbb{R}$ such that $\lambda_i = c_i \lambda$. Pick $b \in \mathbb{Z}^k$ such that for all $i = 1, \ldots, r$ we have

$$| e^{c_i \lambda(b)} - 1 | < \frac{1}{10^6 \| v' \|} < 1.$$

Set $w = v - v'$. Then we get

$$\| b\, v - v \| = \| b\, v' - v' + b\, w - w \| \leq | e^{\lambda(b)} - 1 | \| v' \| + \| b\, w \| + \| w \| \leq 3 \cdot 10^{-6}$$

since $\| b\, w \| \leq D^2 \max_{i=1,\ldots r} e^{c_i \lambda(b)} \| w \| \leq 2 D^2 \| w \| \leq 10^{-6}$. Since $b\, v$ and v lie in $\mathbb{Z}^m$, it follows that $b\, v = v$. Thus b has 1 as an eigenvalue which contradicts the definition of standard action.

Next we will reduce the general case of a $\mathbb{Z}^k$-action to the special case above. Let P be the minimal polynomial of a, and let Q be the greatest common divisor of P and P' over $\mathbb{Z}$. Write $P = Q\, R$, and set $W = Q(a)(\mathbb{R}^m)$. Then W is a defined over the rationals, $\mathbb{Z}^k$-invariant and hence defines a $\mathbb{Z}^k$-invariant closed subtorus of T^m. We claim that $a\, |_W$ has the same set of eigenvalues as a with each eigenvalue occurring with multiplicity 1. In particular, $a\, |_W$ is diagonalizable over $\mathbb{C}$. To see this, first note that the minimal polynomial for $a\, |_W$ is exactly R. Let λ be a root of P of multiplicity l. Then $(x - \lambda)^{l-1}$ divides P and P' and hence Q over $\mathbb{C}$. Thus $x - \lambda$ divides R at most once. Conversely, $x - \lambda$ divides R at least once since otherwise $(x - \lambda)^l$ divides P'. Since $\mathbb{Z}^k$ leaves the one-dimensional eigenspaces of a invariant, it follows that each element of $\mathbb{Z}^k$ is semisimple when restricted to W. Finally, a has one-dimensional unstable manifold in W by construction. Clearly, each element of the $\mathbb{Z}^k$-action restricted to W is invertible. Hence, by the special case above, the $\mathbb{Z}^k$-action on W has Lyapunov exponents which are not multiples of each other.

$\diamond$

Proposition 6.3 *Let α be a standard action of $\mathbb{Z}_+^k$, $k \geq 2$. Suppose one of the following two conditions holds.*

a) *The action contains an Anosov element with one-dimensional unstable distribution.*

b) *For all Lyapunov exponent λ and all $c > 0$, $-c\lambda$ is not a Lyapunov exponent.*

Then we can find elements $a, b \in \mathbb{R}^k$ such that b is regular, $E_b^+ \subset E_a^+$ and a is a generic singular element on the boundary of the Weyl chamber which contains b. In case b), we can pick any generic a in the kernel of λ.

Proof: For **a)** let us pass to the suspension X of the solenoid extension if necessary. Let b be the Anosov element with one-dimensional unstable distribution on the torus. By formula (2) in Section 3, all non-Archimedean Lyapunov exponents of the action on X are nonpositive on b. Hence the unstable distribution E_b^- of b on X is one-dimensional and totally Archimedean. Let λ be the Lyapunov exponent determined by E_b^-. By Lemma 6.2, there is a Lyapunov exponent that is not a multiple of λ. Hence we can pick a Lyapunov exponent μ such that μ is not a multiple of λ and such that the kernel of μ intersects the closure of the Weyl chamber of b. Pick a generic $a \in \mathbb{R}^k$ subject to the condition that $\mu(a) = 0$ and $\lambda(a) \neq 0$. Since a belongs to the closure of the Weyl chamber of b, and $\lambda(a) \neq 0$, $\lambda(a)$ and $\lambda(b)$ are both positive. Since E_b^- is one-dimensional, we find that $E_{b^{-1}}^+ = E_b^- \subset E_a^- = E_{a^{-1}}^+$, as desired.

For **b)** let $H_1, \ldots, H_m$ be the kernels of the Lyapunov exponents. Pick a 2-dimensional plane $\mathcal{P} \subset \mathbb{R}^k$ such that $\mathcal{P} \cap H_i \cap H_j = \{0\}$ whenever $i \neq j$. Note that the Lyapunov exponents restricted to $\mathcal{P}$ are all distinct. Furthermore, the kernels of the restrictions $L_i \stackrel{\text{def}}{=} \mathcal{P} \cap H_i$ coincide if and only if the kernels on $\mathbb{R}^k$ coincide. Indeed, let $\lambda \neq \mu$ be two exponents. If κ and μ have the same kernel then $\kappa = c\mu$ is a multiple of μ with $c \neq 1$. Hence their restrictions to $\mathcal{P}$ are non-trivial multiples. If κ and μ do not have the same kernel then the kernels of their restrictions to $\mathcal{P}$ are distinct by choice of $\mathcal{P}$.

Fix an orientation and a metric on $\mathcal{P}$. Draw all the unit vectors v_i for which there is a Lyapunov exponent β such that $\beta(v_i) = 0$ and β is positive to the right of v_i.

By assumption λ is a Lyapunov exponent such that any other Lyapunov exponent with the same kernel on $\mathcal{P}$ is a positive multiple of λ. Let L be the kernel of λ restricted to $\mathcal{P}$ and pick $a \neq 0 \in L$, $\| a \| = 1$ such that λ is positive to the right of a.

Pick $b \in P$ close to a such that b is to the left of a (which makes sense locally) and such that there is no v_i in the closed cones defined by a, b and $-a$, $-b$ except for a. This is possible since no negative multiple of λ is a Lyapunov exponent. Note that any v_i that lies in the half-plane strictly to the left of b also lies in the half-plane strictly to the left of a. Hence a and b have the same stable distribution, $E_a^+ = E_b^+$. Finally note that we can pick a as any generic element in the kernel of λ by genericity of $\mathcal{P}$. $\diamond$

Combining Theorem 5.1 with Lemma 6.1 and Proposition 6.3, we obtain criteria for rigidity of measures with positive entropy.

Corollary 6.4 *Let α be a standard action of $\mathbb{Z}_+^k$, $k \geq 2$, on a torus T^n.*

a) *Suppose the action contains an Anosov element with one-dimensional unstable distribution. Then μ is either Lebesgue measure or the entropy w.r.t. μ of every element is 0.*

b) *If for all Lyapunov exponents λ and all $c > 0$, $-c\lambda$ is not a Lyapunov exponent then μ is either Lebesgue measure on an invariant rational subtorus or the entropy w.r.t. μ of every element is 0.*

Proof: In both cases pass to the suspension X of the solenoid extension of the $\mathbb{Z}_+^k$-action, if necessary.

In case a) we apply Proposition 6.3 to find elements $a, b \in \mathbb{R}^k$ such that b is Anosov, $E_b^+ \subset E_a^+$ and a is a generic singular element on the boundary of the Weyl chamber which contains b. Therefore we get $E_b^- \supset E_a^0$. Recall from the proof of Proposition 6.3 a) that E_b^- is one-dimensional and totally Archimedean. It follows that $E_b^- = E_a^0$. Also notice that $\xi_a \le \mathcal{W}_a^0$ by Lemma 6.1. This together with the above implies condition (*) of Theorem 5.1 for b^{-1}. Since $E_a^0 \cap E_b^- = E_a^0$ and $\xi_a \le \mathcal{W}_a^0$, condition (**) of Theorem 5.1 holds as well. By Theorem 5.1, either the entropy w.r.t. μ of every element is 0 or μ is Haar measure on a non-zero rational subtorus T^m. Since the unstable foliation of b is dense in T^n and one-dimensional, its leaves are not tangent to T^k. Hence T^m is contained in single stable leaf of b and gets contracted to a point. Since $b(T^m)$ is open and closed in T^m by the nonsingularity of b, $b(T^m) = T^m$. Since T^m gets contracted to a point, T^m is a point which is a contradiction.

In case b), note as before that the unstable foliation on X of a regular element b of $\mathbb{Z}_+^k$ is totally Archimedean. Hence the claim follows from Theorem 5.1, applied to b^{-1}, provided conditions (*) and (**) hold. Since $E_{b^{-1}}^+ = E_b^-$ is a sum of Lyapunov spaces, there are generic singular elements a_i such that

$$E_b^- = \sum_i E_{a_i}^0 \cap E_b^- .$$

Let us check condition (**) for these a_i. By the assumption and Proposition 6.3, there are regular elements b_i with $E_{b_i}^+ \subset E_{a_i}^+$. By Lemma 6.1, we find $\xi_{a_i} \le \mathcal{W}_{a_i}^0 = \xi(E_{a_i}^0) \le \xi(E_{a_i}^0 \cap E_b^-)$. ◇

Let us point out that Examples 3.5 and 3.8 as well as all standard actions of $\mathbb{Z}^{n-1}$ on T^n are covered by the case a) of this Corollary.

7 Symmetric space and twisted examples

Let G be a semisimple connected real Lie group of the noncompact type and of $\mathbb{R}$-rank at least 2. Let A be the connected component of a split Cartan subgroup of G. Suppose Γ is an irreducible lattice in G. Then the action of A on G/Γ is partially hyperbolic. Unlike our results on cocycle rigidity and local differentiable rigidity, our results here do not depend on cocompactness of Γ.

The centralizer $Z(A)$ of A splits as a product $Z(A) = M\,A$ where M is compact. Since A commutes with M, A acts on $N \stackrel{\text{def}}{=} M \setminus G/\Gamma$. This action is called the *Weyl chamber flow*. We will call the Weyl chamber flows as well as the actions on the cover G/Γ or any intermediate cover *standard symmetric space actions*.

Let $\rho : \Gamma \to SL(n, \mathbb{Z})$ be an irreducible representation of Γ such that for at least one $\gamma \in \Gamma$, $\rho(\gamma)$ is Anosov on T^m. Then Γ acts on the n-torus T^n via ρ and hence on $M \setminus G \times T^n$ via

$$\gamma(x, t) = (x\gamma^{-1}, \rho(\gamma)(t)).$$

Let $N \stackrel{\text{def}}{=} M \setminus G \times_\Gamma T^n \stackrel{\text{def}}{=} (M \setminus G \times T^n)/\Gamma$ be the quotient of this action. As the action of A on $M \setminus G \times T^n$ given by $a(x,t) = (ax, t)$ commutes with the Γ-action, it induces an action of A on N. This action is called the *twisted Weyl chamber flow.*

This example generalizes by taking an action on an intermediate cover between G/Γ and $M \setminus G/\Gamma$ as the base space of the twisting. We may also restrict the action of $\mathbb{R}^k$ to a closed subgroup isomorphic to either $\mathbb{R}^m$ or $\mathbb{Z}^m$ with $m \geq 2$ as long as at least one element acts partially hyperbolically with neutral foliation given by the quotient of the MA-orbit foliation. All of these examples are called *standard twisted symmetric space actions.*

Our main result for the symmetric space case is strictly parallel to the toral case. However, due to the symplectic nature of these examples, we do not get as strong applications as described in Section 6.

Theorem 7.1 *Let α be a standard symmetric space $\mathbb{R}^k$-action with $k \geq 2$. Assume that μ is an invariant ergodic measure for α such that there are generic singular elements $a_1, \dots, a_k$ and a regular element $b \in \mathbb{R}^k$ such that*

$$(*) \qquad E_b^+ = \sum_i (E_{a_i}^0 \cap E_b^+)$$

(where the sum need not be direct) and such that

$$(**) \qquad \xi_{a_i} \leq \xi(E_{a_i}^0 \cap E_b^+).$$

Then μ is either Haar measure on a homogeneous real algebraic subspace or every element has 0 entropy w.r.t. μ.

Let us note that there are some examples of symmetric actions on a compact space G/Γ for which there is a subgroup H of the same real rank that intersects Γ in a cocompact lattice in H. Hence the Haar measure on $H/(H \cap \Gamma)$ is invariant for the action of the maximal split Cartan on G/Γ.

Proof: The structure of the proof follows the one for the toral case closely.

It suffices to consider the symmetric space action on G/Γ since stable and unstable manifolds project injectively to any factor. At each step of the argument we arrive at a dichotomy for certain conditional measures. Either they are atomic, and we proceed to the next step or they are Haar on certain homogeneous subspaces inside W_b^+ which is unipotent. In this case, we can apply Ratner's classification of invariant measures for unipotent subgroups of Lie groups [23]. This is the main difference of the argument.

The second difference appears in arriving at the conclusion of Lemma 5.5. Instead of decomposing everything into Lyapunov spaces, we consider the coarser decomposition into sums of Lyapunov spaces with proportional Lyapunov exponents. Note that in some symmetric space examples we can have λ and $2\,\lambda$ as Lyapunov exponents. It follows from the classification of symmetric spaces that this is the only possibility of proportional exponents for the action of the maximal split Cartan subgroup. This decomposition is integrable and its leaves $\boldsymbol{F}(x)$ are the orbits of a unipotent group of nilpotent length 2. Inside each fiber, the fast directions are also integrable and make up the center of the unipotent group in question. Since the center is abelian, we can apply the arguments of Lemma 5.4 and 5.5 to this central foliation. Thus the central conditional measure is either Haar on a subgroup and we can apply Ratner's theorem or it is atomic. In the latter case, we apply the argument of Lemma 5.4 to $\boldsymbol{F}$. We conclude that the conditional measure on $\boldsymbol{F}$ is supported on a graph of a smooth function from a transversal to the central foliation to the central leaves. This smooth function depends measurably on the initial point and its tangent distribution produces an α-invariant subdistribution of $\boldsymbol{F}$ which projects non-trivially to the transversal direction. But such invariant measurable distributions do not exist (cf. Lemma 5.8).

The rest of the argument is identical to the toral case. In fact, since in this case the center foliation coincides with the isometric foliation, Lemma 5.9 is not needed. $\diamond$

Theorem 7.1 implies the statements for the symmetric space case completely analogous to Corollaries 5.2 and 5.3 for the toral case.

The techniques of the proof of the last theorem also yield a partial result in the twisted symmetric space examples.

Theorem 7.2 *Let α be a standard twisted symmetric space $\mathbb{R}^k$-action on M with symmetric space factor M'. Assume that $k \geq 2$ and that μ is an invariant ergodic measure for α such that there are generic singular elements $a_1, \dots, a_k$ and a regular element $b \in \mathbb{R}^k$ satisfying conditions (*) and (**) (cf. Theorem 7.1). Then either every element has zero entropy w.r.t. μ or μ is an extension of a zero entropy invariant ergodic measure on M' by Haar measure along the toral fibers or μ is Haar measure on a homogeneous real algebraic subspace.*

Proof: We may assume again that the base space M' is of the form G/Γ for some cocompact lattice Γ in G. Note that M is foliated by both the fibration over M' as well as the G-orbit foliation. The stable, unstable and neutral foliations for various elements split into the toral part and the G-orbit part. Recall that if the entropy of some element is positive then the entropy of b is also positive (cf. the proof of Theorem 5.1). In this case, the conditional measures of μ on toral or G-orbit parts of the stable foliation of b are non-atomic. In the first case, we proceed as in the proof of Theorem 5.1 and deduce that the family of conditional measures on the toral foliation is Haar on a family of rational subtori. Since the representation ρ of Γ is irreducible, the rational subtori have to coincide with the complete fibers.

In the second case, we proceed as in the proof of Theorem 7.1 and deduce that conditional measures on a certain unipotent subfoliation are Haar. Invoking Ratner's theorem again we find that μ is Haar measure on a homogeneous real algebraic subspace. $\diamond$

We believe that the above argument can be generalized to actions on very general homogeneous and bi-homogeneous spaces.

Finally let us mention that at least in some cases such as compact quotients of $SL(3,\mathbb{R})$ one can obtain some results about arbitrary ergodic invariant measures with positive entropy for some element of the action. One cannot conclude in this case that the measure is Haar on a subspace. In fact, the example due to Rees [24] to which we alluded in the Introduction is a symmetric space flow on $SL(3,\mathbb{R})/\Gamma$ which has a compact invariant homogeneous subspace isomorphic to an S^1-bundle over $SL(2,\mathbb{R})/\Gamma'$. Then any measure on

$SL(2,\mathbb{R})/\Gamma'$ invariant under the maximal Cartan in $SL(2,\mathbb{R})$ (which is an Anosov flow) lifts to an invariant measure for the $\mathbb{R}^2$-action on $SL(3,\mathbb{R})/\Gamma$. Thus we obtain measures of positive entropy which are not Haar. Note that for a regular element b in $\mathbb{R}^2$, the conditional measure on the three-dimensional stable manifold of b is supported on a single one-dimensional fiber corresponding to a Lyapunov subspace in the $SL(2,\mathbb{R})$-direction.

For an arbitrary ergodic $\mathbb{R}^2$-invariant measure μ on a quotient of $SL(3,\mathbb{R})$ with positive entropy for some element we can show that one of the following two possibilities holds: μ is Haar on a homogeneous algebraic submanifold or the conditional measure on the three-dimensional stable foliation for any regular element is supported on a single line. The main extra ingredient in the proof is the use of the non-commutativity of Lyapunov foliations which allows us to transport the conditional measures on a Lyapunov foliation along other Lyapunov foliation inside the same stable foliation. If enough of those measures are non-atomic (e.g. two in the $SL(3,\mathbb{R})$ case) the conditional measures can be shown to possess a translation invariance property similar to the one discussed in the proof of Lemma 4.6.

This local non-integrability of the Lyapunov foliations makes the Weyl chamber flows (which are always symplectic), at least those which come from split simple Lie groups, somewhat more amenable examples than toral symplectic actions where at present we are unable to obtain any results beyond those that follow from Theorem 4.1. Another difficult case is represented by symmetric space examples where the G is the product of rank one simple groups and Γ is an irreducible lattice. For some of those examples, both toral and semi-simple, the picture of the Lyapunov hyperplanes and Weyl chambers looks exactly as for the products of rank one actions where naturally there many ergodic invariant measures of positive entropy. Thus in order to exclude pathological measures with positive entropy one should go beyond the local analysis based on the structure of Lyapunov spaces and use global (probably arithmetical) properties of those actions.

8 Appendix

Here we give an alternative metric structure on the solenoids which is more arithmetical in nature, and is needed to define Lyapunov exponents in the non-Archimedean directions.

For each prime number $p \in \mathbb{Z}_+$, denote by $\mathbb{Q}_p$ and $\mathbb{Z}_p$ the p-adic completions of $\mathbb{Q}$ and $\mathbb{Z}$ respectively and by $|\ |_p$ the p-adic norm. Endow each $\mathbb{Z}_p^m$ with the sup norm. Set $\tilde{I} = \mathbb{R}^m \oplus \bigoplus_p \mathbb{Z}_p^m$ where p ranges over all prime numbers in $\mathbb{Z}$. Note that $\mathbb{Z}_+^k$ acts diagonally on $\tilde{I}$. Also $\mathbb{Z}^m$ acts on $\tilde{I}$ diagonally by integer translations. This action is normalized by the $\mathbb{Z}_+^k$-action. Hence $\mathbb{Z}_+^k$ acts on the quotient $I = \tilde{I}/\mathbb{Z}^m$. Note that I is a fiber bundle over T^m with fiber $\bigoplus_p \mathbb{Z}_p^m$ where p ranges over all primes in $\mathbb{Z}$, and that the $\mathbb{Z}_+^k$-action on I covers α.

Proposition 8.1 *The $\mathbb{Z}_+^k$-action on I extends to a $\mathbb{Z}^k$-action.*

Proof: It suffices to check that each element is surjective and injective. We will slightly abuse notation by identifying elements of $\mathbb{Z}_+^k$ with their images under α. Let $A \in \mathbb{Z}_+^k$. We first show that A is injective on I. Suppose not. Then there is a non-zero element $v = (v_{\mathbb{R}^m}, \ldots, v_p, \ldots) \in \tilde{I}$ where $v_{\mathbb{R}^m}$ is the $\mathbb{R}^m$-coordinate such that there is a $k \in \mathbb{Z}^m$ such that for all primes p we have $Av_p = k$. Hence $v_{\mathbb{R}^m} \in \mathbb{Q}^m$ and v_p is $v_{\mathbb{R}^m}$ embedded into $\mathbb{Q}_p$. Hence the p-adification of $v_{\mathbb{R}^m} \in \mathbb{Q}^m$ is a p-adic integer for all primes p. Therefore $v_{\mathbb{R}^m}$ lies in $\mathbb{Z}^m$.

For surjectivity, let F denote the finite set of primes which divide $\det A$. If $p \notin F$, then $detA$ is a unit in $\mathbb{Z}_p$. By Kramer's rule, A is invertible on $\mathbb{Z}_p^m$ for $p \notin F$. Thus it suffices to show that the image contains $\bigoplus_{p \in F} \mathbb{Z}_p$. Note that A multiplies Haar measure on $\bigoplus_{p \in F} \mathbb{Z}_p^m$ and hence on $\bigoplus_p \mathbb{Z}_p^m$ by the product of $\Pi_{p \in F} \mid \det A \mid_p$. Since this product is $\frac{1}{|\det A|}$, A is an injective measure preserving endomorphism on I. Hence A is surjective. $\diamond$

For each prime p, set

$$M_p^{\mathbb{Q}} = \{x \in \mathbb{Q}_p^m \mid \text{ for all } A \in \mathbb{Z}^k \text{ we have } \mid A\,x \mid_p = \mid x \mid_p\}.$$

Then set $M_p = M_p^{\mathbb{Q}} \cap \mathbb{Z}_p^m$. As noted above, if a prime p does not divide $\det A$, then A is invertible over $\mathbb{Z}_p$. Note that multiplication by a matrix with entries in $\mathbb{Z}_p$ does not increase the p-adic norm of a vector. Thus if p does not divide $\det A$, then A is an isometry on $\mathbb{Q}_p^m$. Hence M_p coincides with $\mathbb{Z}_p^m$ for all but finitely many p.

It is easiest to discuss the properties of M_p, when the matrices are uppertriangular. To this end, we pass to a suitable field extension. This will also prove useful later when we discuss Lyapunov epxonents. We will freely use the basic material on valuations and their completions (cf. for example [18]).

Let K be the number field generated by the eigenvalues of the A_i, and let R be its ring of integers. Then we can uppertriangularize all $A \in \mathbb{Z}_+^K$ simultaneously over K. Let $\boldsymbol{p} \subset R$ be a prime ideal and let $p > 0$ be the prime number that generates $\boldsymbol{p} \cap \mathbb{Z}$. Then $\boldsymbol{p}$ determines a unique absolute value $|\,|_{\boldsymbol{p}}$ such that

$$\mid p \mid_{\boldsymbol{p}} = \frac{1}{p}.$$

The completion $K_{\boldsymbol{p}}$ of K with respect to $|\,|_{\boldsymbol{p}}$ is a finite extension of $\mathbb{Q}_p$ of degree at most the degree of k over $\mathbb{Q}$ (and possibly smaller). Note that $|\,|_{\boldsymbol{p}}$ extends $|\,|_p$.

Lemma 8.2 *The set M_p is a submodule of $\mathbb{Z}_p^m$. Moreover, the quotient $\mathbb{Z}_p^m / M_p$ is a free $\mathbb{Z}_p$-module.*

Proof: Let K and $\boldsymbol{p}$ be as above. Set

$$M_{\boldsymbol{p}}^K = \{x \in K_{\boldsymbol{p}}^m \mid \text{ for all } A \in \mathbb{Z}^k \text{ we have } \mid A\,x \mid_{\boldsymbol{p}} = \mid x \mid_{\boldsymbol{p}}\}.$$

Clearly, $M_p = M_p^K \cap \mathbb{Z}_p^m$. Hence the first claim follows if we show that M_p^K is a vector space. Let $A \in \mathbb{Z}_+^K$. Consider the decomposition of K^m into generalized eigenspaces V_λ of A with eigenvalue λ. Then there is a nilpotent matrix M with integer coefficients such that $A \mid_{V_\lambda} = \lambda\, I + M$ where I denotes the identity matrix. The inverse matrix A^{-1} is of the form $\frac{1}{\lambda} I + N$ where N is another nilpotent matrix with integer coefficients. Suppose that $N^{r+1} = 0$. Note that

$$(\frac{1}{\lambda} I + N)^l = \lambda^{-l}\, I + \begin{pmatrix} l \\ 1 \end{pmatrix} \lambda^{-l+1}\, N + \ldots + \lambda^{-l+r} \begin{pmatrix} l \\ r \end{pmatrix} N^r. \quad (1)$$

Since $A^{-l} \in \mathbb{Z}^k$ acts isometrically on $M_{\boldsymbol{p}}^K$, no vector in K^m with non-zero component in V_λ with $\mid \lambda \mid_{\boldsymbol{p}} \neq 1$ can belong to $M_{\boldsymbol{p}}^K$. Indeed, $\lambda \in R$, and hence $\mid \lambda \mid_{\boldsymbol{p}} < 1$.

Conversely, suppose that $\mid \lambda \mid_{\boldsymbol{p}} = 1$. Then the restrictions to V_λ of both A and A^{-1} are matrices with $R_{\boldsymbol{p}}$-coefficients. Multiplication by such a matrix is automatically norm nonincreasing. Hence $A \mid_{V_\lambda}$ is an isometry.

To summarize, $M_{\boldsymbol{p}}^K$ is precisely the intersection over all $A \in \mathbb{Z}^k$ of the sum of the generalized eigenspaces of A with eigenvalue λ with $\mid \lambda \mid_{\boldsymbol{p}} = 1$. In particular, $M_{\boldsymbol{p}}^K$ is a vector subspace.

To prove the second claim, first note that $\mathbb{Z}_p$ is a principal ideal domain. Hence every finitely generated module over $\mathbb{Z}_p$ is a direct sum of a free module and the submodule of torsion elements. Suppose that $\bar{x} \in \mathbb{Z}_p^m / M_p$ is a torsion element. Pick $a \in \mathbb{Z}_p$ such that $a \neq 0$ and $a\,\bar{x} = 0$. Let $x \in \mathbb{Z}_p^m$ be a representative of $\bar{x}$. Then $a\,x \in M_p$. Hence for all $a \in \mathbb{Z}^k$, $\mid A\,(a\,x) \mid_p = \mid a\,x \mid_p$. Dividing by $\mid a \mid_p$, we see that $x \in M_p$. Hence there are no torsion elements in $\mathbb{Z}_p^m / M_p$. $\diamond$

Set $M = \bigoplus_p M_p$ and $\mathcal{A} = I/M$. By Lemma 3.2, $\mathcal{A}$ fibers over T^m with fiber $\bigoplus_p \mathbb{Z}_p^{m_p}$ and $\mathbb{Z}^k$ acts on it covering the action of $\mathbb{Z}_+^k$ on T^m. We will denote this action by $\hat{\alpha}$. Furnish $\mathcal{A}$ with the product metric ν of the finitely many p-adic metrics and the Euclidean metric.

Let $\mathcal{A}^*$ denote the natural fiber bundle over T^m with fiber $\mathbb{Q}_p^{m_p}$. Note that $\mathbb{Z}^k$ naturally acts on $\mathcal{A}^*$. The fiber splits in a $\mathbb{Z}^k$-invariant way into finitely many subspaces $E_\lambda^{\mathbb{Q}}$ over the various $\mathbb{Q}_p$, where $\lambda : \mathbb{Z}^k \to \mathbb{R}$ is a non-zero linear functional, such that for all $A \in \mathbb{Z}^k$, all $n \in \mathbb{Z}_+$ and all $x \in E_\lambda^{\mathbb{Q}}$ we have

$$\lim \frac{1}{n} \log \mid A^n x \mid_p = \lambda(A).$$

This follows from the multiplicative ergodic theorem (cf. [22, Ch. V, Theorem 2.1]) or simply linear algebra, as we will explain below.

We will call these λ the *non-Archimedean Lyapunov exponents* and the subspaces $E_\lambda^{\mathbb{Q}}$ the corresponding *(rational) Lyapunov spaces.* The intersection $E_\lambda^{\mathbb{Q}} \cap \mathbb{Z}_p^{m_p}$ is a $\mathbb{Z}_p$-submodule of the fiber of $\mathcal{A}$, and in particular a closed subgroup. We will call it a *Lyapunov subspace* or *Lyapunov subgroup.*

Let us now describe the Lyapunov exponents and spaces of $A \in \mathbb{Z}^k$ algebraically. Let us first discuss the case of a single matrix A. The general case for $\mathbb{Z}^k$ follows as usual since the Lyapunov spaces for a single matrix are invariant under all of $\mathbb{Z}^k$ by commutativity and we can then just intersect the Lyapunov spaces for different elements in $\mathbb{Z}^k$.

For a single matrix A, we claim that the Lyapunov exponents are the logarithms of the $\boldsymbol{p}$-adic norms of the eigenvalues of A. Furthermore, given a Lyapunov exponent $\lambda(A)$, the corresponding Lyapunov space is the intersection of the $\mathbb{Q}_p^n$ with the sum of all generalized eigenspaces of A over $K_{\boldsymbol{p}}$ with eigenvalue ν where

$$\log \mid \nu \mid_{\boldsymbol{p}} = \lambda(A).$$

In fact, consider a generalized eigenspace V_ν of A over $K_{\boldsymbol{p}}$. Then $A \mid_{V_\nu} = \nu I + N$ where N is a nilpotent matrix. By formula (1) above, it is clear that the Lyapunov exponent of a vector in V_ν is $\log \mid \nu \mid_{\boldsymbol{p}}$. Recall that the Galois group of $K_{\boldsymbol{p}}$ over $\mathbb{Q}_p$ acts by isometries on $K_{\boldsymbol{p}}$ with repect to $\mid\mid_{\boldsymbol{p}}$ ([19, ch. II, §1]. Hence the sum of the generalized eigenspaces whose eigenvalues have absolute value $e^{\lambda(A)}$ is defined over $\mathbb{Q}_p$, as desired.

Indeed, it is immediate from formula (1) and the discussion of nilpotents in the proof of Lemma 3.2 that the Lyapunov spaces are the intersections of the various eigenspaces of all $A \in \mathbb{Z}^k$ and the Lyapunov exponents are the logarithms of the $\boldsymbol{p}$-adic norms of the eigenvalues of A. It follows that the non-Archimedean Lyapunov exponents on $\mathbb{Z}^k$ have values which are logarithms of rational numbers. Note that these Lyapunov exponents are defined everywhere and independently of any invariant measure. Also note that for all $A \in \mathbb{Z}^k_+$ and all non-Archimedean Lyapunov exponents λ, we get

$$\lambda(A) \le 0. \tag{2}$$

Notice that there are elements of $\mathbb{Z}^k_+$ for which all non-Archimedean Lyapunov exponents are non-zero. The extensions of the generators A_i however may not be amongst such elements. In the Anosov case there are elements such that all Lyapunov exponents, Archimedean as well as non-Archimedean, are non-zero. Dynamically this implies that each point has a neighborhood with a product structure of exponentially expanding and contracting subgroups. For reasons of uniformity of language and ideas, we make the following definition.

Note 8.3 For reasons of uniformity of language and ideas, we will call these expanding and contracting subgroups the *non-Archimedean unstable and stable manifolds* respectively. Note that these "manifolds" just like the non-Archimedean Lyapunov spaces are always subgroups.

Proposition 8.4 *The solenoid S with α^* is isomorphic to $\mathcal{A}$ with $\hat{\alpha}$.*

Proof: Since (S, α^*) is the unique minimal compact group extension of (T^m, α) to a $\mathbb{Z}^k$-action there is an equivariant surjective homomorphism $\pi : \mathcal{A} \to S$. The kernel C of π is a closed subgroup of $\mathcal{A}$, and thus a $\mathbb{Z}$-module under the diagonal embedding of $\mathbb{Z}$ into $\bigoplus_{p\in F} \mathbb{Z}_p$. Since the latter embedding is dense by the Chinese remainder theorem, C is a $\bigoplus_{p\in F} \mathbb{Z}_p$-module. Suppose C is not discrete. Multiplying by a suitable element $(0, \dots, 1, \dots, 0) \in \bigoplus_{p\in F} \mathbb{Z}_p$, C contains a nontrivial $\mathbb{Z}_p$-submodule for some $p \in F$. Let $A \in \mathbb{Z}^k_+$ have all non-Archimedean Lyapunov exponents non-zero, and thus strictly negative. Hence A strictly contracts C, and hence cannot be invertible on S. Thus we see that C is discrete.

As C is a discrete subgroup of a compact group, it is finite. As the fiber is a product of additive subgroups of fields of characteristic 0, C is trivial. $\diamond$

We will also call the solenoid extensions of standard actions by toral endomorphisms and their suspensions standard.

Let us illustrate the construction of the solenoid by some examples.

Example 8.5 Let s and t be positive integers. Consider the action of $\mathbb{Z}_+^2$ on S^1 by multiplication by s and t. In order to construct the solenoid, let $p_1, \ldots, p_l$ be the primes occurring in the prime decomposition of s and t. Let $s = p_1^{k_1} \ldots p_l^{k_l}$ and $t = p_1^{m_1} \ldots p_l^{m_l}$ where the k_i and m_i are nonnegative integers. Then the fiber of S over 0 is $\mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_l}$. There are l non-Archimedean Lyapunov exponents $\lambda_1, \ldots, \lambda_l$ where

$$\lambda_i(x, y) = -(k_i x + m_i y) \log p_i$$

There is one Archimedean Lyapunov exponent of multiplicity one given by

$$\lambda(x, y) = \sum_i (k_i x + m_i y) \log p_i.$$

Example 8.6 Consider the action of $\mathbb{Z}_+^2$ on T^2 generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Both matrices have eigenvalues $1 + 2i$ and $1 - 2i$. They generate the field $K = \mathbb{Q}[i]$ whose ring of integers is the Gaussian integers $\mathbb{Z}[i]$. The principal ideals $\boldsymbol{p}_1 = (1 + 2i)$ and $\boldsymbol{p}_2 = (1 - 2i)$ are prime ideals. We have $\mid 1 + 2i \mid_{\boldsymbol{p}_1} = 1/5$ while $\mid 1 - 2i \mid_{\boldsymbol{p}_1} = 1$, and similarly $\mid 1 + 2i \mid_{\boldsymbol{p}_2} = 1$ while $\mid 1 - 2i \mid_{\boldsymbol{p}_2} = 1/5$. Note that the $\boldsymbol{p}_1$- and $\boldsymbol{p}_2$-adifications of $\mathbb{Z}[i]$ are just $\mathbb{Z}_5$ since $\mathbb{Z}_5$ contains a square root of -1 (cf. e.g. [16]).

The fiber of the solenoid S over 0 is $\mathbb{Z}_5^2$. There are two non-Archimedean Lyapunov exponents λ_1 and λ_2, each with multiplicity 1 given by

$$\lambda_1(x, y) = -x \log 5 \qquad \text{and} \qquad \lambda_2(x, y) = -y \log 5$$

where $(x, y) \in \mathbb{Z}^2$. Also note that there is one Archimedean Lyapunov exponent of multiplicity 2 given by

$$\lambda(x, y) = (x + y)\frac{\log 5}{2}.$$

Example 8.7 Let $\mathbb{Z}_+^2$ act on T^2 by the generators

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 12 & 8 \\ 8 & 4 \end{pmatrix}.$$

This is a faithful action since for any m, n, $\det A^m B^n = (-5)^m(-16)^n \neq 1$ unless both m and n are 0. In this case, one toral direction is contracted by both A and B although both of them are non-invertible. It is a useful exercise to describe the non-Archimedean Lyapunov exponents explicitly for this example.

Not every automorphism of a solenoid which covers T^m comes from an endomorphism of T^m. To illustrate this let us consider the following example.

Example 8.8 Let S be the solenoid whose dual group is $(\mathbb{Z}(p_1, \ldots, p_l))^m$ where $p_1, \ldots, p_l$ are distinct prime integers. This solenoid covers T^m with the fiber $\mathbb{Z}_{p_1}^m \times \ldots \times \mathbb{Z}_{p_l}^m$. Any $m \times m$ matrix with rational entries whose decomposition into prime factors contains only powers of $p_1, \ldots, p_l$ determines an automorphism of S.

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