Corrections to
Invariant Measures for Higher Rank Hyperbolic Abelian Actions

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The proofs of Theorems 5.1 and 7.1 of [2] contain a gap. We will show below how to close it under some suitable additional assumptions in these theorems and their corollaries. We will assume the notation of [2] throughout. In particular, μ is a measure invariant and ergodic under an R^k-action α. Let us first explain the gap. Both theorems are proved by establishing a dichotomy for the conditional measures of μ along the intersection of suitable stable manifolds. They were either atomic or invariant under suitable translation or unipotent subgroups U. Atomicity eventually led to 0 entropy. Invariance of the conditional measures showed invariance of μ under U. Then we claimed that μ is algebraic using unique ergodicity of the translation subgroup on a rational subtorus or Ratner's theorem respectively (cf. [2, Lemma 5.7]). This conclusion however only holds for the U-ergodic components of μ which may not equal μ. In fact, in the toral case, the R^k-action may have a 0-entropy factor such that the conditional measures along the fibers are Haar measures along a foliation by rational subtori. Since invariant measures with 0 entropy have not been classified, we cannot conclude algebraicity of the total measure μ at this time. In the toral case, the existence of zero entropy factors turns out to be precisely the obstruction to our methods. The case of Weyl chamber flows is somewhat different as the "Haar" direction of the measure may not be integrable. In this case, we need to use additional information coming from the semisimplicity of the ambient Lie group to arrive at the versions of Theorem 7.1 presented below.

The toral case: Here we discuss the corrections to Theorem 5.1 of [2] and its corollaries. We also indicate a slight generalization of the theorem using condition $(\mathcal{R})$ introduced by A. Starkov in [4] for a semigroup α of endomorphisms (automorphisms) of T^m, isomorphic to Z_+^k (resp. Z^k).

$(\mathcal{R})$: The action α contains a semigroup ρ, isomorphic to Z_+^2, which consists of ergodic endomorphisms.

Let α and α' be two actions of Z_+^k by endomorphisms of T^m and $T^{m'}$ correspondingly. Call α' *an algebraic factor* of α if there exists an epimorphism $h : T^m \to T^{m'}$ such that $h \circ \alpha = \alpha' \circ h$. The action α is called *completely irreducible* if any non–trivial algebraic factor has finite fibers. We will say that α' is a *rank–one factor* of α if a subsemigroup of finite index of $\alpha'(Z_+^k)$ consists of powers of a single map. Extending the arguments from [4] which deal with the invertible case (actions by automorphisms) one sees that condition $(\mathcal{R})$ is equivalent to

$(\mathcal{R}')$: The action α does not possess non–trivial rank one algebraic factors.

The next result replaces our main theorem, Theorem 5.1, from [2].

Theorem 5.1' *Let α be a R^k-action with $k \geq 2$ induced from an action by toral endomorphisms satisfying condition $(\mathcal{R})$. Assume that μ is an ergodic invariant measure for α such that there are generic singular elements $a_1, \dots, a_k$ and a regular element $b \in R^k$ with E_b^+ totally Archimedean such that*

C1: $E_b^+ = \sum_i (E^0_{a_i} \cap E_b^+)$ *(where the sum need not be direct) and*

C2: $\xi_{a_i} \leq \xi(E^0_{a_i} \cap E_b^+)$.

Then the measure μ_{T^m} is an extension of a zero–entropy measure in an algebraic factor of smaller dimension with Haar conditional measures in the fibers.

The proof of Theorem 5.1 in [2]) is based on a sequence of lemmata (5.4-5.10). The lemmata dealing with conditional measures (5.4, 5.5, 5.6, 5.9, and 5.10) are correct and continue to hold under condition $(\mathcal{R})$ without any changes in the proofs. Lemma 5.7 which is not specific for actions by endomorphisms is obviously false without an ergodicity assumption; hence one cannot derive Lemma 5.8 which is directly based on it. Instead of these two lemmata Lemma 5.8' below holds.

Let $\boldsymbol{F}(x) \subset W_a^I$ be any a-invariant Archimedean subfoliation of W_a^I. Let $\mu_x^{\mathbf{F}}$ denote the system of conditional measures determined by $\boldsymbol{F}$ normalized by the requirement that $\mu_x^{\boldsymbol{F}}(B_1^{\boldsymbol{F}}(x)) = 1$ for all x in the support of μ where $B_1^{\boldsymbol{F}}(x)$ is the ball of radius 1 w.r.t. the induced metric on $\boldsymbol{F}$.

Lemma 5.8' *Let $\boldsymbol{F}$ be the foliation $W_a^I \cap W_b^+$. For μ-a.e. x, the conditional measure $\mu_x^{\mathbf{F}}$ is atomic unless μ_{T^m} is the extension of an invariant measure in an algebraic factor of smaller dimension with Haar measures in the fibers.*

Proof: Denote by S_x the support of $\mu_x^{\mathbf{F}}$. By Lemmata 5.6 and 5.7 of [2], for μ-a.e. x, the S_x are affine subspaces, and $\mu_x^{\mathbf{F}}$ is Haar measure on S_x. By ergodicity of the action the subspaces S_x for a.e. x are parallel. In particular, the conditional measures are either atomic or non–atomic a.e. Assume the second possibility. The fact that the conditional measures are Haar is equivalent to the measure μ_{T^m} being invariant under the subgroup of translation determined by those spaces and hence under the closure G of that subgroup. The orbits of G are parallel rational subtori and the partition into these orbits is α–invariant. Hence it determines an algebraic factor of α of smaller dimension. By unique ergodicity of minimal linear foliations on the torus we conclude that the conditional measures are Haar measures along the fibers. ◇

Now the proof of Theorem 5.1' proceeds as in [2] with the following modification. Once we arrive at the assumption of Lemma 5.8' there is a dichotomy. If the conditionals are atomic we proceed to use Lemmata 5.9 and 5.10. Otherwise we obtain an algebraic factor of smaller dimension. If the factor–measure has zero entropy for all elements of α, we are done. Otherwise we may assume that in the factor $\alpha(b)$ still has positive entropy and repeat the argument. Note that condition $(\mathcal{R})$ is inherited by any factor. We will arrive at a factor of the factor and so on. Since at every step the dimension of the factor drops this process has to stop, thus producing a factor with zero entropy. It is also clear by induction that the conditionals are in fact Haar measures on the fibers. So if, for example, the ultimate factor turns out to be trivial then the original measure is Haar. ◇

Several consequences of Theorem 5.1 have to be changed to allow for 0 entropy factors as well. We will just list them here. Corollary 5.3 as stated does not follow as weak mixing for the Z^k-action on the torus does not necessarily imply weak mixing of the R^k-action.

Corollary 5.2' *Let α be a R^k-action with $k \geq 2$ induced from an action by toral endomorphisms satisfying condition $(\mathcal{R})$. Assume that μ is an α-invariant measure such that every one-parameter subgroup is ergodic or equivalently, that μ is weakly mixing w.r.t. α. Then the measure μ_{T^m} is an extension of a zero–entropy measure in an algebraic factor of smaller dimension with Haar conditional measures in the fibers.*

Proposition 6.3 a) and Corollary 6.4 a) remain correct as stated since Anosov actions with one-dimension expanding foliations cannot have algebraic factors. For Proposition 6.3 b) and Corollary 6.4 b) we need to assume that α is completely irreducible.

Weyl chamber flows: Next we discuss standard symmetric space actions and in particular Weyl chamber flows. We let G be a semisimple connected real algebraic group of real rank at least 2 and without compact factors. Let Γ be an irreducible lattice in G, A a split Cartan subgroup of G, and μ an A-invariant ergodic measure on G/Γ. Even though Γ is irreducible, μ may be a product of a 0 entropy measure for an irreducible action of a higher rank subgroup of A on one factor with a Haar measure on the other factor. Thus the hypothetical existence of non-algebraic measures of 0 entropy forces us to modify the claim of Theorem 7.1. We will indicate two different conditions sufficient to fill the gap, yielding Theorems 7.1 A and 7.1 B below. We first recall the state of affairs from [2] and prove some facts needed for both theorems.

As in Theorem 7.1 of [2] let us assume that α is a standard symmetric space R^k-action with $k \geq 2$, that μ is an α-invariant ergodic measure such that there are generic singular elements $a_1, \ldots, a_k$ and a regular element $b \in R^k$ such that

C1: $E_b^+ = \sum_i (E_{a_i}^0 \cap E_b^+)$ (where the sum need not be direct) and

C2: $\xi_{a_i} \leq \xi(E_{a_i}^0 \cap E_b^+)$.

We will assume in addition that G is real algebraic. Note that it suffices to consider symmetric space actions on G/Γ. Similar to the toral case, analyzing conditional measures of μ along suitable foliations, the argument in [2] either shows that μ has 0 entropy or that there is a nontrivial unipotent subgroup of G which leaves μ invariant. Suppose the latter. Let L be the connected component of the identity of the stabilizer of μ, and let $M \subset L$ be the maximal Lie subgroup of L generated by unipotent subgroups of L. Note that M is normal in L. By the above, M is nontrivial. Let μ_x denote the ergodic components of μ w.r.t. M. Since M is normalized by $A \subset L$, we see that

$$a\mu_x = \mu_{a\,x}.$$

By Ratner's theorem [3, Corollary C] applied to M, each μ_x is algebraic, i.e. μ_x is Haar measure on some closed orbit $H_x(x)$ of some algebraic subgroup H_x of G which contains M. Consider the measurable function ϕ from G/Γ to the space of algebraic subgroups $\mathcal{H}$ of G given by $x \mapsto H_x$. Since $a\mu_x = \mu_{a\,x}$, a maps the support of μ_x to that of $\mu_{a\,x}$. Hence $a\,H_x\,x = H_{a\,x}\,a\,x$, and thus $\phi(a\,x) = a\,\phi(x)\,a^{-1}$. Since the quotient of $\mathcal{H}$ modulo conjugacy is countably separated and μ is A-ergodic, ϕ takes values in the same conjugacy class μ-a.e.. Thus we can think of ϕ as a map into G/N where N is the normalizer of some H_x in G. Note that N is algebraic. By [1, 3.1 Corollary], applied to $M = G/N$, ϕ and thus $H_x = H$ is μ-a.e. constant. In particular, $H_x = H_{a\,x} = a\,H_x\,a^{-1}$ for μ-a.e. x and $a \in A$. Hence A normalizes H. Since $a\,\mu_x = \mu_x$ and the μ_x are Haar measures for H, it follows that the adjoint action of A on H preserves the volume. Since A is a split Cartan, this for´ces the unipotent radical of H to be trivial. Thus H is reductive. In fact, M then is the product of all noncompact simple factors of H, and thus M is semisimple. Finally note that the μ_x are the ergodic components of μ w.r.t. H.

Theorem 7.1 A *Let α be a standard symmetric space action of $A = R^k$ for $k \geq 2$. Suppose μ is a weakly mixing measure for A on G/Γ. Then μ is either Haar measure on a homogeneous real algebraic subspace or every element has 0 entropy w.r.t. μ.*

Proof: Since every one-parameter subgroup of A acts ergodically w.r.t. μ, the conditions C1 and C2 above are automatically satisfied and we may use the discussion before the theorem. Since

H is reductive, $A \cap H$ is a split Cartan of H and thus $A \cap H \neq 1$. Then $A \cap H$ contains a one-parameter subgroup and hence μ is ergodic w.r.t. $A \cap H$ since μ is weakly mixing. Thus the H-ergodic components μ_x equal μ, and μ is algebraic by Ratner's theorem. $\diamond$

The next theorem resembles Theorem 7.1 of [2] most closely. Note however that both the hypotheses are stronger and the conclusion weaker.

Theorem 7.1 B *Let α be a standard symmetric space R^k-action with $k \geq 2$. Let μ be an invariant ergodic measure for α with the following property:*

Assume that for any maximal nontrivial intersection $\bigcap_{i=1\ldots r} \mathcal{W}^-_{b_i}$ of stable manifolds of elements $b_1, \ldots, b_r \in R^k$ there is an element $a \in R^k$ such that for all $x \in M$, $\bigcap_{i=1\ldots r} E^-_{b_i}(x) \subset E^0_a(x)$ and such that a.e. leaf of the intersection $\bigcap_{i=1\ldots r} \mathcal{W}^-_{b_i}$ is contained in an ergodic component of μ for the one-parameter subgroup $t\,a$ of R^k.

Then μ is either Haar measure on a homogeneous real algebraic subspace or some *element has 0 entropy w.r.t. μ.*

Proof: The hypothesis on maximal intersections of stable manifolds in the theorem implies conditions C1 and C2 above. Thus we may assume the results of the discussion before Theorem 7.1 A. If $A \subset H$, then $\mu = \mu_x$ since μ is A-ergodic by assumption. In particular μ is algebraic. Thus we will assume that $A \cap H$ is a proper subspace of A. Since H is reductive and normalized by A, $A \cap H$ is a split Cartan in H. Since the (restricted) roots of H are spanned by at most $\dim A \cap H$ many simple roots and $\dim A \cap H < \dim A$, there is an element $c \in A$, $c \neq 1$ which lies in the kernel of all the (restricted) roots of H. We claim that the metric entropy $h_\mu(c) = 0$. Note that c has a strong stable foliation $\mathcal{W}^+_c$ and a weak unstable foliation defined everywhere by Lie theory. They are transverse foliations and satisfy the assumptions of Proposition 4.1 of [2]. Hence it suffices to show that the conditional measures along $\mathcal{W}^+_c$ are atomic. For this we decompose the strong stable space E^+_c as a sum of maximal intersections of stable spaces with E^+_c. Using the main argument of [2], we deduce that either all conditional measures of μ along the corresponding subfoliations are atomic or that μ is invariant under some unipotent U tangent to $\mathcal{W}^+_c$. Then $U \subset L$ and thus $U \subset M \subset H$ which contradicts our choice of c since $\mathcal{W}^+_c$ is transversal to the H-orbits. Thus all conditional measures of μ along the relevant subfoliations are atomic. As in [2], Proposition 5.10 of [2] now implies that the conditional measures along $\mathcal{W}^+_c$ are atomic. $\diamond$

Finally, let us remark that we obtain similar results for twisted Weyl chamber flows replacing Theorem 7.2 from [2].

Theorem 7.2' *Let α be a standard symmetric space R^k-action on M with symmetric space factor M'. Assume that $k \geq 2$ and that μ is an invariant weakly mixing measure for α for which conditions C1 and C2 hold. Then either every element has 0 entropy w.r.t. μ or μ is an extension of a zero entropy invariant ergodic measure on M' by Haar measure along the toral fibers or an extension of a Haar measure on M' by zero entropy measures along the toral fibers or μ is Haar measure on a homogeneous real algebraic subspace.*

References

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