Harmonic analysis in rigidity theory

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Contents

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1 Introduction

This article surveys the use of harmonic analysis in the study of rigidity properties of discrete subgroups of Lie groups, actions of such on manifolds and related phenomena in geometry and dynamics. Let me call this circle of ideas *rigidity theory* for short. Harmonic analysis has most often come into play in the guise of the representation theory of a group, such as the automorphism group of a system. I will concentrate on this avenue in this survey. There are certainly other ways in which harmonic analysis enters the subject. For example, harmonic functions on manifolds of nonpositive curvature, the harmonic measures on boundaries of such spaces and the theory of harmonic maps play an important role in rigidity theory. I will mention some of these developments.

Rigidity theory became established as an important field of research during the last three decades. The first rigidity results date back to about 1960 when A. Selberg, E. Calabi and A. Vesentini and later A. Weil discovered various deformation, infinitesimal and perturbation rigidity theorems for certain discrete subgroups of Lie groups. At about the same time, M. Berger proved his purely geometric 1/4-pinching rigidity theorem for positively curved manifolds [184, 28, 27, 205, 206]. But the most important and influential early result was achieved by G. D. Mostow in 1968. In proving his celebrated Strong Rigidity Theorem, Mostow not only provided a global version of the earlier local results, but also introduced a battery of novel ideas and tools from topology, differential and conformal geometry, group theory, ergodic theory, and harmonic analysis. Mostow's results were the catalyst for a host of diverse developments in the ensuing years. Mostow himself generalized his strong rigidity theorem to locally symmetric spaces in 1973 [146]. In 1974, a second major breakthrough occurred when G. A. Margulis discovered his ingenious superrigidity and arithmeticity theorems for higher rank locally symmetric spaces [127, 129]. All along, H. Furstenberg had been developing his probabilistic approach to rigidity and introduced the idea of boundaries of groups [66, 67, 69]. R. J. Zimmer has been building his important deep program of studying actions of "large" groups on manifolds since 1979 [220, 223, 225]. M. Ratner's work over the last decade has provided a deep and fundamental analysis of the rigidity of horocycle flows and unipotent actions. [169, 170, 172, 173]. Rigidity theory blossomed throughout the 1980s and early 1990s. Further major contributors were: W. Ballmann, Y. Benoist, M. Brin, K. Burns, K. Corlette, P. Eberlein, P. Foulon, E. Ghys, M. Gromov, A. Katok, U. Hamenstadt, S. Hurder, F. Labourie, N. Mok, P. Pansu, R. Schoen, Y.T. Siu, R. Spatzier and others.

Let me briefly outline the paper. Section 2 presents a synopsis of rigidity theory. At the heart of this section lies Mostow's theorem and a summary of its proof. I also discuss Margulis' superrigidity and arithmeticity theorems and outline further major developments in rigidity theory. In the next three sections, I describe various tools from harmonic analysis and how they are used in rigidity theory. Each of these sections addresses a particular tool. In section 3, I explain Mautner's phenomenon, the ergodicity of homogeneous flows and its application to Mostow's theorem. Mautner's phenomenon is closely related to the vanishing of matrix coefficients and their rate of decay. This as well as two applications of the rate of decay to rigidity constitute the remainder of Section 3. In Section 4, I introduce amenable actions, and use them to set up the first step of the proof of Margulis' superrigidity theorem. Some other applications to the rigidity of group actions are given as well. Section 5 introduces Kazhdan's property (T). It has been enormously successful, especially in the study of group actions. I include a selection of these topics. In Section 6, I report on a variety of other applications of harmonic analysis to rigidity theory.

Finally, let me thank S. Adams and my two referees for their many valuable comments and suggestions.

2 A Synopsis of Rigidity theory

2.1 Early results

It is probably futile to attempt a general definition of rigidity. However, a common feature of a rigidity theorem is that some fairly weak conditions suddenly and surprisingly force strong consequences. Rigidity theorems usually come in one of the following forms:

a) a deformation or perturbation of a system is equivalent to the original system,

b) a system preserving a weak structure is forced to preserve a strong structure,

c) a weak isomorphism between two objects implies a strong isomorphism.

Already the early history of rigidity theory has theorems of all three types. To begin with, A. Selberg proved in 1960 that a discrete subgroup Γ of $SL(n, \mathbb{R})$ with $n \geq 3$ and $SL(n, \mathbb{R})/\Gamma$ compact cannot be continuously deformed except by inner automorphisms of $SL(n, \mathbb{R})$ [184]. Selberg's method extended to the other classical groups of real rank at least 2. Around the same time, E. Calabi and A. Vesentini proved that the complex structure of a compact quotient of a bounded symmetric domain is rigid under deformations [28]. Slightly later, E. Calabi proved deformation rigidity of compact hyperbolic n-spaceforms for $n \geq 3$ [27]. Remarkably, there are smooth non-isometric families of compact hyperbolic 2-space forms, as was long known at the time. A. Weil generalized Selberg's and Calabi's results to all semisimple groups without compact or three-dimensional factors in 1962 [205, 206].

In 1961, M. Berger proved that a Riemannian manifold of curvature between 1 and 4 which is not homeomorphic to a sphere is isometric to a symmetric space. While this is a classical example of a rigidity theorem (of the second type), its proof is entirely differential geometric. More importantly, the source of rigidity in this case is ellipticity rather than hyperbolicity, unlike most of the examples we will discuss (cf. [79] for a discussion of non-hyperbolic rigidity phenomena).

There are some other early rigidity results. J. Wolf found in 1962 that if G/Γ is compact, then the rank of G is determined by Γ [208]. This was later generalized to non-uniform lattices by G. Prasad and M. S. Raghunathan in [160]. Using probabilistic considerations, H. Furstenberg showed in 1967 that a lattice in $SL(n, \mathbb{R})$ cannot be a lattice in $SO(n, 1)$ [66]. Y. Matsushima and S. Murakami as well as M. S. Raghunathan obtained vanishing theorems for the cohomology of representations of uniform lattices [136, 166, 168, 167].

2.2 Mostow's strong rigidity theorems

In 1968, Mostow proved his celebrated strong rigidity theorem for hyperbolic space forms. It is the prime example of a rigidity theorem of the third type. We will discuss it in detail in the remainder of this section. Let us first fix some notations. We will always call compact manifolds without boundary *closed*. Given a Riemannian manifold M, we denote its universal cover by $\tilde{M}$, its tangent bundle by TM and its unit tangent bundle by SM. Denote by γ_v the unique unit speed geodesic tangent to v at 0. Recall that the *geodesic flow* $g_t : SM \to SM$ takes a unit vector v to $\gamma_v'(t)$, the vector tangent to γ_v at time t. Recall that the sectional curvature function K is a real valued function on the set of 2-planes in TM.

Theorem 2.1 [Strong Rigidity Theorem, Mostow, 1968] *Suppose M and N are closed manifolds of constant sectional curvature -1. Assume M has dimension at least 3. If there is an isomorphism $\psi : \pi_1(M) \to \pi_1(N)$ then M and N are isometric.*

For surfaces, this theorem fails completely. In fact, there is a $6g - 6$-dimensional space, the so-called *moduli space*, that parametrizes all metrics of constant curvature up to diffeomorphism on a surface of genus $g \geq 2$. Here I will just outline the so-called "pair of pants" construction to exhibit a non-trivial continuous family of constant curvature metrics on a closed surface of genus $g \geq 2$. This construction can be used to describe the moduli space completely [26].

Elementary considerations in hyperbolic geometry show that, given any positive real numbers l_1, l_2 and l_3, there exists a hexagon in the hyperbolic plane $\mathcal{H}^2$ with geodesic edges $e_1, \ldots, e_6$ such that the length of e_{2i} is l_i for $i = 1, 2$ and 3 [26]. By gluing two such hexagons along e_1, e_3 and e_5, we obtain a metric of constant curvature -1 on a two-sphere minus three open disjoint disks, a "pair of pants". Note that the boundary curves of this pair of pants are geodesics. Gluing two such pairs of pants along edges of equal length, we obtain a surface Σ of genus 2 with a metric of constant curvature -1 which has closed geodesics of length $2\,l_1, 2\,l_2$ and $2\,l_2$. Recall that the lengths of closed geodesics with respect to a fixed metric form a countable set of numbers. Hence, varying the l_i continuously, we obtain non-isometric metrics on a surface of genus 2.

Similar constructions can be made for any closed surface of genus g at least 2. In fact, such a surface is obtained by gluing $2\,(g-1)$ pairs of pants. Note that the rotation by which two boundary geodesics of two pairs of pants are glued together gives another parameter of the construction. Careful analysis shows that these $6g-6$ length and rotation parameters actually parametrize the set of all metrics of constant curvature -1 on Σ [26].

Theorem 2.1 is completely equivalent to a theorem about subgroups of Lie groups. Given any locally compact group G, we call a discrete subgroup Γ of G a *lattice* if the Haar measure μ on G/Γ is finite and G-invariant. We call a lattice *uniform* or *cocompact* if G/Γ is compact. Otherwise we call Γ *non-uniform*.

Let $SO(n,1)$ be the group of $n+1 \times n+1$ real matrices which preserve the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \mathbf{x_0 y_0} - \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{x_i y_i}$$

on $\mathbb{R}^{n+1}$. Let $SO_0(n,1)$ denote the connected component of the identity of $SO(n,1)$. Then $SO_0(n,1)$ acts transitively and effectively on the sheet of the hyperboloid

$$\mathcal{H}^n \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^{\mathbf{n+1}} \mid \mathbf{x_0} > \mathbf{0}, \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{1}\}$$

through $(1, 0, \ldots, 0)$. The restriction of the quadratic form $\langle\,,\,\rangle$ to the tangent bundle of $\mathcal{H}^n$ is positive definite, and the resulting Riemannian metric has constant curvature -1. We call $\mathcal{H}^n$ the *real hyperbolic space* of dimension n. Note that the isotropy group of $(1, 0, \ldots, 0)$ consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ . & \\ . & \text{A} \\ . & \\ 0 & \end{pmatrix}$$

where A is in $SO(n)$, i.e. A is an orthogonal $n \times n$ matrix matrix of determinant 1. Thus $\mathcal{H}^n$ is the homogeneous space $SO(n) \setminus SO_0(n,1)$, and the metric $\langle\,,\,\rangle$ on $\mathcal{H}^n$ can be described in terms of the unique $SO_0(n,1)$-invariant quadratic form on $SO_0(n,1)$, the so-called Cartan-Killing form. It is well-known that any complete simply connected manifold of constant curvature -1 is isometric to the real hyperbolic space

$\mathcal{H}^n$. Thus the universal cover of any closed manifold M of constant negative curvature -1 is $\mathcal{H}^n$. Therefore we can write M as $\mathcal{H}^n/\Gamma$ or $SO(n)\setminus SO_0(n,1)\,/\,\Gamma$ where Γ is a group of isometries of $\mathcal{H}^n$. It is elementary that $\mathcal{H}^n/\Gamma$ is isometric to $\mathcal{H}^n/\Gamma'$ for some discrete subgroup Γ' of $SO_0(n,1)$ if and only if Γ and Γ' are conjugate in $SO_0(n,1)$. It follows that the geometric version of Mostow's rigidity theorem above is completely equivalent to the following group theoretic rigidity theorem. These remarks also explain the relationship between Selberg's, Calabi's and Weil's results mentioned above.

Theorem 2.2 [Strong Rigidity Theorem, Algebraic Form, Mostow, 1968] *Let Γ be a cocompact lattice in $SO_0(n,1)$, and $n \geq 3$. Let $\psi : \Gamma \to SO_0(m,1)$ for some m be an injective homomorphism such that $\psi(\Gamma)$ is a cocompact lattice in $SO_0(m,1)$. Then $m = n$ and ψ is the restriction of an inner automorphism of $SO_0(n,1)$.*

G. Prasad generalized this theorem to non-uniform lattices in 1973 [159]. M. Gromov proved the theorem in a totally different and completely geometric way in 1979 [80, 196]. Mostow's original proof has also been streamlined and extended to other discrete subgroups of $SO_0(m,1)$, latest by Ivanov in 1993 [198, 8, 103].

Let us give an outline of Mostow's proof. For more details we refer to [196, 81]. Harmonic analysis enters in step 3 below. As we point out there, while historically significant, one can easily substitute a geometric argument (e.g. Hopf's argument) for the harmonic analysis used here. However, the same ideas from harmonic analysis have since proved useful for other rigidity theorems.

Outline of Mostow's proof: Set $\Gamma = \pi_1(M)$. The argument proceeds in several steps.

Step 1: Call a map ϕ between two metric spaces (X,d) and (X',d') a *quasi-isometry* if there are constants $C > 0$ and $E > 0$ such that

$$\frac{1}{C} d(x,y) - E < d'(\phi(x),\phi(y)) < C\, d(x,y) + E.$$

The isomorphism $\psi : \Gamma \to \pi_1(N)$ gives rise to a quasi-isometry $\phi : \tilde{M} \to \tilde{N}$ between the universal covers of M and N. Moreover, ϕ is Γ-equivariant. This means that for all $\gamma \in \Gamma$ and all $x \in \tilde{M}$, we have

$$\phi(\gamma\, x) = \psi(\gamma)\, \phi(x)$$

where γ and $\psi(\gamma)$ act on $\tilde{M}$ and $\tilde{N}$ by deck transformations.

Essentially one constructs ϕ as follows. First one maps a fundamental domain of Γ in $\tilde{M}$ to a fundamental domain of $\pi_1(N)$ in $\tilde{N}$ in an arbitrary way. This map extends uniquely to a Γ-equivariant map on $\tilde{M}$. Note that ϕ need not be continuous, as there maybe discontinuities along the boundary of the fundamental domain. Note that Mostow in his original argument showed how to make Φ into a homeomorphism.

Step 2: Compactify $\mathcal{H}^n$ as follows. Call two geodesic rays *asymptotic* if they are a finite distance apart. The set of asymptote classes is called the *sphere at infinity* S^{n-1}. One can topologize S^{n-1} as well as $\mathcal{H}^n \cup S^{n-1}$ by the so-called "cone topology" [14]. The sphere at infinity then is homeomorphic to a standard sphere.

In the unit disk model of hyperbolic space, this amounts to compactifying the open ball $\mathcal{H}^n$ in Euclidean space to the closed ball. While the above construction works for any simply-connected complete manifold of nonpositive curvature, the sphere at infinity of $\mathcal{H}^n$ has a differentiable and a conformal structure, as one can see from the unit disk model. Standard Lebesgue measure induces a measure class on S^{n-1}. One can abstractly describe this as a *visual measure*. This means, that the measure of a set X in the sphere at infinity is given by the measure of the unit tangent vectors v at a fixed point $p \in \mathcal{H}^n$ such that the asymptote class of the geodesic ray determined by v lies in X.

Any action on $\mathcal{H}^n$ by isometries naturally extends to a continuous action on S^{n-1}. Furthermore, the extension is conformal and preserves the measure class on S^{n-1}.

Thus both $\tilde{M}$ and $\tilde{N}$ are compactified by their spheres at infinity S^{n-1} and S^{m-1}. The quasi-isometric image of a geodesic ray in $\tilde{M}$ lies a finite distance from a (unique) geodesic ray. This crucial fact follows from the the negative curvature of $\tilde{N}$. Thus the quasi-isometry $\phi : \tilde{M} \to \tilde{N}$ extends to a map of the spheres at infinity $\bar{\phi} : S^{n-1} \to S^{m-1}$. By the Γ-equivariance of ϕ, $\bar{\phi}$ is also Γ-equivariant.

One also shows that $\bar{\phi}$ is *quasi-conformal*. This means that the images of all small spheres have bounded *distortion*. Here the distortion of a set is the infimum of the fractions of the radii of circumscribed and inscribed balls. It follows from analysis, that $\bar{\phi}$ is differentiable almost everywhere, and the derivatives and their inverses have uniformly bounded norm. Hence we see that $n \leq m$, and by symmetry that $m = n$. Thus we can identify $\tilde{M}$ and $\tilde{N}$ with $\mathcal{H}^n$.

Step 3: The action of Γ on S^{n-1} is ergodic. Mostow deduced this directly from the Mautner phenomenon in representation theory (cf. Section 3.1). Ergodicity of the Γ-action also follows from the ergodicity of the geodesic flow. The latter can be seen either again using the Mautner phenomenon or via the Hopf argument (cf. Section 3.1).

Using the quasi-conformality and the ergodicity of Γ on S^{m-1}, one can show that the distortion of $\bar{\phi}$ is constant. Then one argues that the distortion is 1. Therefore $\bar{\phi}$ is conformal.

Step 4: Any conformal map of S^{n-1} extends to a unique isometry of $\mathcal{H}^n$. By the Γ-equivariance of $\bar{\phi}$, the isometric extension $\tilde{\phi}$ of $\bar{\phi}$ is Γ-equivariant. Hence $\tilde{\phi}$ descends to an isometry of M to N. This finishes the proof of Mostow's theorem. ◇

Step 2 above established the existence of a conjugacy $\bar{\phi}$ between the two actions of Γ on two spheres at infinity, and that $\bar{\phi}$ is quasi-conformal. The remainder of the proof then concludes that a quasi-conformal conjugacy is conformal, and the two embeddings of Γ into $SO(n, 1)$ are conjugate. We can interpret this last part of Mostow's proof as a dynamical rigidity theorem, about conjugacy of actions of groups of isometries of hyperbolic space on the sphere at infinity. More generally one can ask what kind of conjugacies between two actions of a group on the sphere at infinity of hyperbolic space imply conjugacy of the groups? Continuity alone of course is not sufficient. R. Bowen and D. Sullivan showed around 1980 that absolute continuity of the conjugacy suffices [23, 193]. In 1985, P. Tukia further minimized the amount of differentiability needed [198]. For simplicity, let us state a version of his theorem for cocompact lattices in $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$.

Theorem 2.3 [Tukia, 1985] *Assume Γ_1 and Γ_2 are cocompact lattices in $PSL(2, \mathbb{R})$. Suppose Γ_1 and Γ_2 are isomorphic and that the actions of Γ_1 and Γ_2 on the circle at infinity are conjugate by a homeomorphism which has a non-zero derivative in at least one point. Then Γ_1 and Γ_2 are conjugate in $PSL(2, \mathbb{R})$.*

This avenue was further pursued in [6, 7, 8]. In a recent preprint, N. Ivanov gave a remarkably simple proof of Tukia's theorem and generalized it to higher derivatives and dimensions [103].

Mostow himself generalized his theorem from hyperbolic spaces to compact locally symmetric spaces of the noncompact type in 1973 [146]. To exclude two-dimensional hyperbolic space, he assumed that the space does not have closed one or two dimensional geodesic subspaces which are direct factors locally. Any locally symmetric space can be written as a double quotient $K \setminus G/\Gamma$ where G is the isometry group of the universal cover, K is a maximal compact subgroup of G and Γ is a torsion-free uniform lattice in G. Specific examples are hyperbolic spaces or $SO(n) \setminus SL(n, \mathbb{R})/\Gamma$ where Γ is a uniform lattice in $SL(n, \mathbb{R})$. As for hyperbolic spaces, there is a geometric and a group theoretic version of Mostow's strong rigidity theorem for locally symmetric spaces [146]. Let us only present the algebraic version in detail.

Theorem 2.4 [Strong Rigidity Theorem, Mostow, 1973] *Let G and H be connected semisimple real Lie groups without compact factors and trivial center. Let Γ be a lattice in G, and assume that there is no factor G' of G, isomorphic to $PSL(2, \mathbb{R})$ which is closed modulo Γ, i.e. such that $\Gamma G'$ is closed in G.*

Suppose $\psi : \Gamma \to H$ is an injective homomorphism such that $\psi(\Gamma)$ is a lattice in H. Then ψ extends to a smooth isomorphism $G \to H$.

Mostow proved this theorem for uniform lattices. Strong rigidity for certain non-uniform arithmetic lattices had been obtained by algebraic and arithmetic means in 1967 by H. Bass, J. Milnor and J. P. Serre and also by M. S. Raghunathan [15, 168]. For general non-uniform lattices, strong rigidity was established by G. Prasad in [159]. While substantially more complicated, Mostow's ideas in the proof for locally symmetric spaces are similar to the constant curvature case. Again one argues that there is a quasi-isometry between the universal covers which extends to the sphere at infinity. For symmetric spaces with negative curvature, there is a generalized conformal structure on the sphere at infinity, and one can proceed as in the constant curvature case. If the symmetric space has some 0 curvature, the sphere at infinity carries the structure of a so-called *spherical Tits building* [146, 197]. They are generalizations of classical projective geometry. Tits had invented them to discuss all semisimple algebraic groups from a geometric point of view in a uniform way. Mostow then used Tits' extension of the fundamental theorem of projective geometry to Tits geometries to see that the map induced by the quasi-isometry on the sphere at infinity is induced by an isomorphism of the ambient Lie groups.

2.3 Margulis' superrigidity theorem

The assumption in Mostow's strong rigidity theorem that the image $\psi(\Gamma)$ be a lattice is quite restrictive. G. A. Margulis achieved a breakthrough in 1974 when he determined the homomorphisms of lattices in higher rank semisimple Lie groups G into Lie groups H under only mild assumptions [129, 127]. Higher rank refers to the *real rank* of G. Geometrically, this is the maximal dimension of a totally geodesic flat subspace of the symmetric space $K\backslash G$. Algebraically, one can define it in terms of the maximal dimension of a so-called *split Cartan* subgroup (cf. Section 3.2). Let us call a lattice Γ in G *reducible* if there are connected infinite normal subgroups G' and G'' in G such that $G' \cap G''$ is central in G, $G' \cdot G'' = G$, and the subgroup $(\Gamma \cap G') \cdot (\Gamma \cap G'')$ has finite index in Γ. Otherwise we call Γ *irreducible*. Note that any lattice in a simple Lie group is irreducible. Let us note here that any connected simple real algebraic group always contains both uniform and non-uniform lattices [22, 21, 165]. While Margulis originally only considered semisimple range groups, he later developed the following much refined version of his superrigidity theorem [127, 129].

Theorem 2.5 [Superrigidity Theorem, Margulis, 1974] *Let G be a connected semisimple Lie group with finite center, real rank at least 2, and without compact factors. Let Γ be an irreducible lattice in G. Let $\psi : \Gamma \to H$ be a homomorphism of Γ into a real algebraic group H. Then the Zariski closure of the image $\psi(\Gamma)$ is semisimple. Suppose that $\psi(\Gamma)$ is not relatively compact, that the Zariski closure of $\psi(\Gamma)$ has trivial center and does not have non-trivial compact factors. Then ψ extends uniquely to a continuous homomorphism from G to H.*

Again, the main point in the proof is to construct a Γ-equivariant map between certain "boundaries" of G and H, and to show that such maps are automatically smooth, and give rise to an extension of ψ to G. While the original proof did not directly involve harmonic analysis, R. J. Zimmer in 1979 used amenability to construct such a boundary map [216, 220]. This also allowed him to generalize superrigidity to cocycles of actions of semisimple groups. We will discuss these developments in Section 4.

Margulis refined his theorem to homomorphisms taking values in algebraic groups over arbitrary and in particular p-adic fields [127, Ch. VII, Theorem 6.5]. As a corollary, Margulis obtained his famed arithmeticity theorem [127, 129]. Briefly, an *arithmetic group* is any group commensurable with the points of integers of an algebraic group defined over the integers. Arithmetic groups are always lattices [165]. A typical example is $SL(n,\mathbb{Z})$ in $SL(n,\mathbb{R})$. Note though that $SL(n,\mathbb{Z})$ is a non-uniform lattice.

Theorem 2.6 [Arithmeticity Theorem, Margulis, 1974] *Let* Γ *and* G *be as in Theorem 2.5. Then* Γ *is arithmetic.*

Superrigidity and arithmeticity theorems fail for lattices in many real rank 1 groups [126, 201, 82, 147, 148, 150, 45]. Using differential geometric methods from the theory of harmonic maps, K. Corlette in the Archimedean case, and M. Gromov and R. Schoen in the non-Archimedean case extended Margulis' results to certain rank one spaces.

Theorem 2.7 [Corlette, 1990, Gromov-Schoen, 1992] *Let* Γ *be a lattice in* $G = Sp(n, 1)$*,* $n \geq 2$*, and the exceptional real rank 1 group* F_4^{-20}*. Then any homomorphism* $\Gamma \to GL_n$ *into the general linear group over a local field extends unless the image of* Γ *is precompact. Furthermore,* Γ *is arithmetic.*

The idea in the proof is very different from Mostow's scheme. One first finds a Γ-equivariant harmonic map between the associated globally symmetric spaces, using a general existence theorem originally due to J. Eells and J. Sampson [49]. Then one shows that this map is totally geodesic using a "Bochner formula" for a certain 4-tensor. This idea goes back to Y.-T. Siu's strong rigidity theorem for Kähler manifolds of negative bisectional curvature [186]. In the quaternionic case, the Bochner formula is so strong that one obtains superrigidity rather than just strong rigidity.

N. Mok, Y.-T. Siu and S.-K. Yeung recently found Bochner formulas for suitable 4-tensors for all higher rank globally symmetric spaces, quaternionic hyperbolic spaces and the Cayley plane [141, 140, 139]. Thus they reprove and extend both Margulis' and Corlette's superrigidity theorems using harmonic maps in a fairly unified geometric way. In part, this work is a nonlinear version of the ideas of Y. Matsushima and S. Murakami [136]. Similar results were obtained by J. Jost and S.-T. Yau [104]. These methods also yield a fair amount of information for complex hyperbolic spaces and other Kähler manifolds, as was shown by W. Goldman and J. Millson and later J. Carlson and D. Toledo [74, 29].

Another high point is Margulis' finiteness theorem. It asserts that all normal subgroups of an irreducible lattice Γ in a semisimple Lie group of the noncompact type and higher rank are either central or have finite index. In the course of the proof, Margulis also determined the measurable quotients of the natural Γ-action on the "boundaries" of G, e.g. projective spaces [127].

2.4 Further developments

We will now give a brief outline of further important developments in rigidity theory during the last two decades. In later sections we will discuss those advances that are connected with harmonic analysis in more detail.

Actions of semisimple groups and their lattices

Due to the Margulis' superrigidity theorems, one essentially understands the finite dimensional representations of an irreducible lattice in a semisimple Lie group of higher rank. Naturally one asks if other representations of such a group are similarly restricted.

Infinite dimensional unitary representations of such a lattice in general are quite unwieldy. In fact, E. Thoma showed in 1964 that a discrete group is type I if and only if it has an abelian subgroup of finite index [195]. On the positive side however, such representations satisfy some restrictions, such as Kazhdan's property (cf. Section 5) or the recent results of M. Cowling and T. Steger on restrictions of unitary representations of the ambient semisimple group to the lattice (cf. Section 3.4) [119, 36].

The "finite dimensional nonlinear representation theory" of such groups, especially the study of smooth actions on manifolds, lies in between and is much more restricted. R. J. Zimmer started their study in 1979, when he generalized Margulis' superrigidity theorems to cocycles over finite measure preserving actions of semisimple groups and their lattices.

Given an action of a group G on a measure space M with measure μ and another group H, a measurable map $\beta : G \times M \to H$ is called a *measurable cocycle* if it satisfies the cocycle identity $\beta(g_1 g_2, m) = \beta(g_1, g_2 m)\,\beta(g_2, m)$ for μ-a.e. $m \in M$ [220]. The simplest cocycles are the constant cocycles, i.e. those μ-a.e. constant in M. They correspond to homomorphisms $G \to H$. As another example, suppose G acts differentiably on a manifold M by α. Let μ be a Lebesgue measure on M and choose a measurable framing of M. Then the derivatives of $g \in G$ acting on M determine elements in $GL(n, \mathbb{R})$ at every point of M. Due to the chain rule, this defines a cocycle, the so-called *derivative cocycle* of α.

Call two cocycles β and β^* *measurably cohomologous* if there exists a measurable function $P : M \to H$, called a *measurable coboundary*, such that $\beta^*(a, x) = P(ax)^{-1}\,\beta(a, x)\,P(x)$ for all $a \in G$ and a.e. $x \in M$. For example, a change of the measurable framing of a manifold determines a cohomologous derivative cocycle. Thus often we are only interested in the cohomology class of the cocycle.

Let us call a finite measure preserving ergodic action *irreducible* if it remains ergodic when restricted to any non-trivial normal subgroup of G.

Theorem 2.8 [Cocycle Superrigidity, Zimmer, 1980] *Let G be a connected semisimple real algebraic group without compact factors and of rank at least 2. Let α be an irreducible ergodic finite measure preserving action of G on a measure space M. Let H be a connected non-compact simple real algebraic group. Suppose $\beta : G \times M \to H$ is a measurable cocycle which is not cohomologous to a cocycle taking values in an algebraic subgroup L of H. Then β is cohomologous to a constant cocycle.*

Amenability plays an essential role in the proof of this theorem (cf. Section 4). There are also versions of this theorem for p-adic and complex semisimple groups as range groups. It is not known if Theorem 2.8 holds for range groups which are not semisimple. However, suppose G acts by principal bundle automorphisms on a principal H-bundle $P \to M$ over a compact manifold M. Trivialize the bundle measurably. Then the bundle automorphisms give rise to a cocycle $\beta : G \times M \to H$. Zimmer extended Theorem 2.8 to these cocycles in 1990 [226]. This extends earlier work by G. Stuck and Zimmer himself and gives a full generalization of Margulis' superrigidity theorem for these cocycles [191, 225].

Using harmonic maps, K. Corlette and R. Zimmer found a quaternionic version of the cocycle superrigidity theorem under somewhat stronger assumptions [33]. M. Cowling and R. Zimmer had earlier developped certain rigidity statements for lattices in $Sp(n, 1)$ and their actions using von Neumann algebra techniques [37] [37].

The cocycle superrigidity theorems impose severe restrictions on the derivative cocycle of a smooth volume preserving action, especially when the action preserves further geometric structures. This was shown by Zimmer and others in a remarkable series of papers (cf. e.g. [225, 223]). By now the study of such actions is an important area in rigidity theory. We will be discussing various aspects of this below.

Riemannian geometry

The last decade has brought several important developments on the differential geometric side of rigidity. M. Gromov showed in 1981 that any compact non-positively curved manifold with fundamental group isomorphic to that of a compact locally symmetric space of the non-compact type of higher rank is isometric to the symmetric space [14]. This was also shown for locally reducible spaces by P. Eberlein in [47]. Gromov's proof explores the Tits geometry of such a space, and is quite close to Mostow's ideas. W. Ballmann, M. Brin, P. Eberlein and I introduced a purely differential geometric notion of rank in 1984 [12, 13]. Given a complete Riemannian manifold M, its *rank* is the minimal dimension of the space of parallel Jacobi fields along any bi-infinite geodesic. If M has nonpositive sectional curvature, M has higher rank if and only if every geodesic is contained in a totally geodesic flat 2-plane. We established a structure theory for such spaces in [12, 13]. These developments culminated in the rank-rigidity theorem by W. Ballmann and independently K. Burns and myself [11, 25, 14].

Theorem 2.9 [Ballmann, Burns-Spatzier, 1985] *A closed locally irreducible higher rank Riemannian manifold of non-positive curvature is locally symmetric.*

The proof by Burns and myself is again inspired by Mostow's approach. The use of the dynamics of the geodesic flow proved to be a major ingredient in both our and Ballmann's proofs.

P. Eberlein and J. Heber generalized the rank rigidity theorem to certain noncompact manifolds, in particular spaces of finite volume [48]. Heber established the theorem for homogeneous manifolds of nonpositive curvature [91]. S. Adams extended the theorem to leaves of foliations of closed manifolds by manifolds of nonpositive curvature and higher rank [4]. The general case of a noncompact manifold remains open,

The quaternionic hyperbolic spaces and the Cayley upper half plane play a special role within manifolds of strictly negative curvature. For these spaces, P. Pansu extended Mostow's work on quasi-isometries in a remarkable way in 1989 [156].

Theorem 2.10 [Pansu, 1989] *Any quasi-isometry of a quaternionic hyperbolic space or the Cayley upper half plane is a finite distance from an isometry.*

The real and complex hyperbolic cases are very different. For $\mathcal{H}^n$ for example, any diffeomorphism of S^{n-1} extends to a quasi-isometry of $\mathcal{H}^n$. It is not known if Pansu's theorem extends to the irreducible higher rank symmetric spaces of nonpositive curvature.

Berger's 1/4-pinching theorem found its counterpart for compact quotients of rank one non-constant curvature symmetric spaces in the works of U. Hamenstadt, L. Hernandez, and S. T. Yau, F. Zheng [87, 95, 209].

Theorem 2.11 [Hamenstadt, Hernandez, Yau-Zheng, 1990's] *Any* $1/4$*-pinched metric on a closed locally symmetric space of negative, nonconstant curvature is symmetric.*

This theorem was first proved in dimension 4 by M. Ville via an inequality between the signature and the Euler characteristic [200].

The latter papers use harmonic maps in an essential way. They had been introduced to rigidity theory by Y. T. Siu in his generalization of Mostow's rigidity theorem to certain Kähler manifolds in 1980 [186]. J. Sampson improved Siu's argument to show that harmonic maps from a Kähler manifold to a manifold with nonpositive curvature operator are pluriharmonic [181]. J. Carlson and D. Toledo used these results systematically to study harmonic maps from Kähler manifolds into locally symmetric spaces [29]. This approach to rigidity culminated in the extensions of the superrigidity and arithmeticity theorems to quaternionic spaces by Corlette and Gromov and Schoen [32, 83], and the general geometric proof by Mok, Siu and Yeung [141].

T. Farrell and L. Jones proved topological analogues of Mostow's rigidity theorem for manifolds of variable negative and nonpositive curvature over the last decade. While metric rigidity clearly fails in negative curvature, they established rigidity of the homeomorphism type [52, 53, 54, 55].

Theorem 2.12 [Farrell-Jones, 1980's] *Any two closed non-positively curved manifolds of dimension bigger than 5 with isomorphic fundamental group are homeomorphic.*

Farrell and Jones obtained similar rigidity results for manifolds of nonpositive curvature [54]. Further generalizations to polyhedra of negative curvature were obtained by B. Hu [100]. Farrell and Jones also disproved rigidity of the diffeomorphism type.

Theorem 2.13 [Farrell-Jones, 1990's] *In dimension 7 and up, there are closed manifolds of negative sectional curvature which are homeomorphic but not diffeomorphic to a real (or complex) hyperbolic space.*

The idea of their construction is to glue an exotic sphere into a closed hyperbolic space M. The new space is homeomorphic but not diffeomorphic to M. Then they explicitly construct a metric of negative sectional curvature on the new space. In the complex hyperbolic space, the metrics obtained have sectional curvatures between -1 and $-4-\delta$ for arbitrarily small $\delta > 0$. This complements Theorem 2.11. P. Ontaneda recently obtained 6-dimensional closed manifolds which are homeomorphic but not diffeomorphic to a closed real hyperbolic manifold by a totally different technique [152]. Note that none of these examples is diffeomorphic to a locally symmetric space by Mostow's rigidity theorem. There are also negatively curved closed manifolds which are not even homotopy equivalent to a locally symmetric space. G. Mostow and Y. T. Siu constructed a certain 4-dimensional Kähler manifold of this type [151]. Later M. Gromov and W. Thurston found a rather flexible construction of such spaces using ramified coverings of closed real hyperbolic spaces [84]. In particular, they found such spaces with sectional curvatures arbitrarily close to -1.

Dynamics of amenable groups

One of the most profound developments in dynamical rigidity is M. Ratner's work on the horocycle flow. Recall that the horocycle flow on the unit tangent bundle of a surface is the flow along the stable manifolds parametrized by arc length. Group theoretically, it is the flow of the one parameter group generated by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Ratner showed in 1980 that if the horocycle flows of two surfaces of constant curvature -1 are measurably isomorphic, then the surfaces are isometric [169]. She later determined the factors and joinings of the horocycle flows. Various generalizations to higher dimensional locally symmetric spaces and variable curvature were established by J. Feldman and D. Ornstein, C. Croke, P. Otal, D. Witte, L. Flaminio and myself [56, 59, 207, 61].

In a breakthrough in 1990, M. Ratner proved a very general theorem about invariant sets and measures for unipotent flows on homogeneous spaces which subsumes her previous works mentioned above. Recall that a subgroup U of a connected Lie group G is called *unipotent* if for each $u \in U$, $\mathrm{Ad}\,(u)$ is a unipotent automorphism of the Lie algebra of G. The horocycle flow is an example of a unipotent subgroup of $SL(2,\mathbb{R})$. M. S. Raghunathan had conjectured that all closed invariant subsets of a unipotent flow on a homogeneous space G/Γ, Γ a lattice in G, are algebraic. Ratner showed this via the following measure theoretic generalization of Raghunathan's conjecture.

Theorem 2.14 [Ratner, 1990] *Let U be a unipotent subgroup of G, and Γ a lattice in G. Then any invariant probability measure μ for the action of U on G/Γ is a Haar measure on a closed homogeneous subspace of G/Γ.*

Special cases of this theorem had been obtained by H. Furstenberg, W. Parry, and S. G. Dani [64, 68, 158, 43, 44]. G. A. Margulis had shown Raghunathan's topological conjecture for $SL(3,\mathbb{R})$ and used it to prove the Davenport conjecture in number theory [133]. The theorem was generalized to p-adic groups by M. Ratner and independently by G. Margulis and G. Tomanov [175, 174, 134, 135].

Ratner's theorem has strong implications for actions of semisimple groups. R. J. Zimmer and later Zimmer and A. Lubotzky used it to find topological obstructions to the existence of actions of semisimple groups on a manifold [229, 123]. The following is a typical example of their work.

Theorem 2.15 [Lubotzky-Zimmer, 1993] *Suppose a connected simple non-compact Lie group G of $\mathbb{R}$-rank at least 2 acts real analytically on a closed manifold M, preserving a real analytic connection and a volume. Then any faithful linear representation of $\pi_1(M)$ in $GL(m,\mathbb{C})$ contains a lattice in a Lie group which locally contains G.*

A. Katok and the author used Ratner's theorem to give a partial classification of invariant measures for homogeneous Anosov $\mathbb{R}^k$-actions for $k \geq 2$ [117].

The classification of Anosov flows and diffeomorphisms with smooth stable foliation was another major theme in the last decade. E. Ghys started this investigation in the three-dimensional case in 1987 [71]. Recall the relevant definitions. Let M be a closed manifold with a fixed Riemannian norm$\| \ \|$. Then a C^∞-flow ϕ_t on M is called *Anosov* if there is a splitting of the tangent bundle $TM = E^s \oplus E^u \oplus \frac{d}{dt}\phi_t$ of M into invariant subbundles E^s and E^u and the flow direction $\frac{d}{dt}\phi_t$ and there exist constants $C > 0$ and $m > 0$ such that for all $v \in E^s$ and $t > 0$ (($v \in E^u$ and $t < 0$ respectively)

$$\| \, d\phi_t(v) \, \| \leq C \, e^{-m|t|}.$$

The distributions E^s and E^u are called the *stable* and *unstable* distributions respectively. They are integrable, and define the *stable* and *unstable foliations* of ϕ_t. While the individual leaves of the stable and unstable foliations are C^∞, the dependence of the leaves on the initial point in general is only Hölder. Y. Benoist, P. Foulon and F. Labourie obtained the following description of Anosov flows with smooth stable foliations [16].

Theorem 2.16 [Benoist-Foulon-Labourie, 1990] *Suppose a C^∞ Anosov flow ϕ_t preserves a contact structure. If the stable foliation of ϕ_t is C^∞, then a finite cover of ϕ_t is C^∞-conjugate to a time change of a geodesic flow of a locally symmetric space of negative curvature.*

They also determined which time changes can occur. Furthermore, if ϕ_t is the geodesic flow of a manifold of negative curvature, then ϕ_t itself is C^∞-conjugate to a geodesic flow of a locally symmetric space of negative curvature. This generalizes earlier work of M. Kanai, R. Feres and A. Katok who prove this theorem for geodesic flows of manifolds with sufficiently pinched negative curvature [106, 57, 58]. There are similar theorems for Anosov diffeomorphisms [17].

P. Foulon and F. Labourie applied the techniques used in Theorem 2.16 to negatively curved asymptotically harmonic manifolds, i.e. manifolds whose horospheres have constant mean curvature [63]. C. Yue then combined these techniques with his work on the Margulis' function to resolve the Green conjecture in odd dimensions [211].

Theorem 2.17 [Foulon-Labourie, Yue, 92] *Let M be a closed Riemannian manifold of negative curvature such that the mean curvature of a horosphere at v only depends on the footpoint of v. Then M is asymptotically harmonic. Its geodesic flow is C^∞-conjugate to that of a locally symmetric space of negative curvature. Furthermore, if $\dim M$ is odd, then M has constant curvature.*

Earlier C. Yue had shown that closed Riemannian manifolds of negative curvature for which the harmonic measure equals the Liouville measure are asymptotically harmonic [210]. Combining this with Theorem 2.17 shows that odd dimensional manifolds of this kind have constant negative curvature. This affirms a particular instance of the Sullivan conjecture.

Quite recently, rigidity phenomena associated with higher rank have been observed in the dynamics of small groups. These phenomena first surfaced in the study of the rigidity of the standard action of $SL(n, \mathbb{Z})$ on the n-torus by S. Hurder, A. Katok, J. Lewis and R. J. Zimmer where the local C^∞-rigidity of the action of $n-1$ commuting Anosov automorphisms of T^n proved crucial [101, 110, 111, 112]. Recall that an action α of a discrete group G is called *locally C^k-rigid* if any perturbation of the action which is C^1-close on a generating set is C^k-conjugate to the original action. This definition does not generalize well to Lie groups. Indeed, we may always compose α with an automorphism ρ of G, close to the identity, to get a perturbation of α. Thus let us call an action of an arbitrary group G *locally C^k-rigid* if any perturbation of the action which is C^1-close on a generating set is C^k-conjugate to α composed with an automorphism of G.

Call an action of a group *Anosov* if one element of the group acts normally hyperbolically to the orbit foliation (cf Section 3.3 for more details). In 1992, A. Katok and the author introduced a certain class of homogeneous Anosov actions of $\mathbb{R}^k$ and $\mathbb{Z}^k$, called *standard Anosov actions* [114]. A typical example is the action of the diagonal group in $SL(n,\mathbb{R})$ on $SL(n,\mathbb{R})/\Gamma$ by left translations where Γ is a uniform lattice in $SL(n,\mathbb{R})$ and $n > 2$. Other examples are generated by commuting Anosov toral automorphisms. All the standard actions are actions by $\mathbb{R}^k$ or $\mathbb{Z}^k$ for $k \geq 2$. Conjecturally, all homogeneous actions of $\mathbb{R}^k$ are standard unless they have a rank one factor, i.e. a factor on which a hyperplane in $\mathbb{R}^k$ acts trivially. We established a local rigidity theorem for such actions [114, 115, 116, 118].

Theorem 2.18 [Katok-Spatzier, 1992] *The standard Anosov $\mathbb{R}^k$- or $\mathbb{Z}^k$-actions are C^∞-rigid if $k \geq 2$.*

Much earlier, in 1976, R. Sacksteder had established infinitesimal rigidity for expanding toral endomorphisms [180].

In the proof of Theorem 2.18 we need to show that certain cocycles are cohomologous to constant cocycles. Unlike in Zimmer's superrigidity theorem however, all our cocycles and coboundaries are C^∞ or Hölder and take values in abelian groups.

Theorem 2.19 [Katok-Spatzier, 1992] *Every $\mathbb{R}$-valued C^∞ (Hölder)-cocycle over a standard $\mathbb{R}^k$-Anosov action is C^∞ (Hölder)-conjugate to a constant cocycle if $k \geq 2$.*

This last theorem uses harmonic analysis in an essential way, as I will explain in detail in Section 3.3. Later we applied our techniques to obtain the local C^∞-rigidity of projective actions of lattices [118]. We also classified a certain class of invariant measures for the standard Anosov actions [115]. A. Katok and K. Schmidt extended Theorem 2.19 to automorphisms of compact abelian groups other than tori [113]. A. and S. Katok obtained vanishing results for higher cohomologies of higher rank abelian automorphism groups of the torus [109].

There is very little known about general Anosov actions by higher rank abelian groups. They seem to be quite rare. J. Palis and J. C. Yoccoz have shown that generically, Anosov diffeomorphisms on tori only commute with their own powers [154, 155].

3 Mautner's Phenomenon and Asymptotics of Matrix Coefficients

Rigidity problems in geometry, group theory and dynamics are often closely related, as our discussion of Mostow's theorem has shown. It is natural to try to apply harmonic analysis in the guise of representation theory as a tool. I will now describe various techniques and ideas from harmonic analysis that have proved useful in the study of rigidity.

E. Hopf discovered the ergodicity of the geodesic flow of a manifold of constant negative curvature in 1939 by a geometric argument [97]. The relation between such geodesic flows and certain one-parameter subgroups of $SL(2,\mathbb{R})$ was only discovered in S. V. Fomin's and I. M. Gelfand's article [62] in 1952. There, Fomin and Gelfand identified the geodesic flow of a compact surface $\mathcal{H}/\Gamma$ of constant negative curvature with the homogeneous flow induced by the one-parameter subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $SL(2,\mathbb{R})/\Gamma$. Then they used the representation theory of $SL(2,\mathbb{R})$ to determine the spectrum of the geodesic flow. In the same paper, they also initiated the general study of homogeneous flows. O. Parasyuk used these ideas in 1953 to discuss the horocycle flow of a compact surface. In 1957, F. Mautner determined the ergodic components of the geodesic flow of a compact or finite volume locally symmetric space M of the non-compact type [137]. Like Fomin and Gelfand, he modeled the geodesic flow of such a space by certain homogeneous flows. He then proved their ergodicity by analyzing isotropy groups of unitary representations. C. C. Moore discovered a beautiful and simple criterion for the ergodicity of a homogeneous flow on a homogeneous space of a semisimple group in 1966 [142]. His analysis was based on Mautner's ideas on isotropy groups.

Homogeneous flows and their ergodicity were further investigated by L. Auslander, L. Green, F. Hahn, C. C. Moore, J. Brezin and others [10, 9, 142, 190, 41, 42, 143, 24].

In this section we will first describe Mautner's and Moore's results and give a proof in a simple case. Then I will introduce matrix coefficients, discuss their rough asymptotics and derive Moore's theorem from them. In the remainder of this section, I will describe precise results on the decay of matrix coefficients of smooth vectors. This fine asymptotics of matrix coefficients has been applied to rigidity in dynamics in three ways.

1. R. J. Zimmer showed that higher rank lattices cannot act ergodically on sufficiently low dimensional manifolds, preserving a volume, provided the action is "N-distal" [225];
2. A. Katok and the author showed that the first cohomology of certain hyperbolic abelian actions is trivial;
3. M. Cowling and T. Steger investigated restrictions of unitary representations of the ambient Lie group to lattices. A. Iozzi applied their results to equivariant maps between lattice actions.

I will discuss applications 2) and 3) in some detail. For the first application I refer to Zimmer's survey [225].

3.1 The Mautner-Moore results

Let us start with some definitions. Let G denote a connected semisimple real Lie group without compact factors. The prime example of such a group is $SL(n,\mathbb{R})$. In most of our discussion, the reader may substitute $SL(n,\mathbb{R})$ for a general G. I will illustrate the majority of the results and definitions by that example. If Γ is a lattice in G and $\{a_t\} \subset G$ a one-parameter subgroup, call the flow by the left translations of $\{a_t\}$ on G/Γ *homogeneous*. If G is unimodular, then $\{a_t\}$ leaves Haar measure μ on G/Γ invariant. If μ is ergodic for $\{a_t\}$, we call the homogeneous flow ergodic.

The next theorem is Moore's extension of Mautner's fundamental work on the ergodicity of the geodesic flow of a symmetric space [137].

Theorem 3.1 [Moore, 1966] *Let Γ be a lattice in G, and $\{a_t\} \subset G$ a one-parameter subgroup. Then the homogeneous flow by $\{a_t\}$ on G/Γ is ergodic if and only if $\{a_t\}$ is not precompact in G.*

Note that these homogeneous flows encompass geodesic and horocycle flows as well as various frame flows and other extensions. These flows exhibit a variety of geometric behaviors which make a geometric treatment of the ergodicity of such flows rather difficult.

Mautner's discovery was based on a fixed point phenomenon for unitary representations of G. For simplicity, let us consider the case when $G = SL(2,\mathbb{R})$ and

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let

$$h_s^+ = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Then g_t and h_s^+ satisfy the commutation relation

$$g_{-t}\, h_s^+\, g_t = h^+_{s\,e^{-2t}}.$$

Geometrically, one can interpret g_t as a geodesic flow and h_s^+ as its horocycle flow. Then the commutation relation above just expresses the standard exponential contraction of the stable horospheres by the geodesic flow.

Now suppose that ρ is a continuous unitary representation of $SL(2,\mathbb{R})$ on a Hilbert space V, and that g_t fixes a vector $v \in V$. The commutation relation above implies for fixed s that $g_{-t}\, h_s^+\, g_t \to 1$ as $t \to \infty$. Hence we see that

$$\| \rho(h_s^+)v - v \| = \| \rho(h_s^+)\rho(g_t)v - v \| = \| \rho(g_{-t}\, h_s^+\, g_t)v - v \| \to 0$$

as $t \to \infty$. Thus v is fixed by h_t.

A similar argument applies to the opposite horocycle flow given by the matrices

$$h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Since g_t, h_s^+ and h_s^- generate $SL(2,\mathbb{R})$, we see that v is fixed by all of $SL(2,\mathbb{R})$. Thus we have shown that a vector of a unitary representation of $SL(2,\mathbb{R})$ fixed by g_t is already fixed by all of $SL(2,\mathbb{R})$. This is the so-called *Mautner phenomenon*. It extends to various one-parameter subgroups of more general groups G [127, Chapter II, 3].

Now let us apply Moore's theorem to the unitary representation on $L^2(G/\Gamma)$, induced from the action by left translations on a homogeneous space G/Γ. As the one-parameter group g_t is not precompact, we see that a g_t-invariant L^2-function is already G-invariant, and hence constant. Thus the homogeneous flow induced by g_t is ergodic.

Let M be the surface $\mathcal{H}^2/\Gamma$. Then its unit tangent bundle can be identified with $SL(2,\mathbb{R})/\Gamma$ as $SL(2,\mathbb{R})$ acts transitively on points and directions of $\mathcal{H}^2$. The geodesic flow of M becomes the homogeneous flow of g_t on $SL(2,\mathbb{R})/\Gamma$. Thus the argument above proves the ergodicity of the geodesic flow of a surface of constant curvature. E. Hopf gave a much more geometric argument for the ergodicity of the geodesic flow in constant negative curvature [97]. The spirit of it however is the same. Hopf's main idea is to show that a function invariant under the geodesic flow is constant along stable and also unstable manifolds. The latter are the orbits of h_s^+ and h_s^-. The stable manifolds are contracted exponentially fast by the geodesic flow. This is the geometric interpretation of the commutation relation between g_t and h_s^+, and also the key to Hopf's argument.

For $G = SO(n,1)$ with $n > 2$, matters get more complicated, since the geodesic flow itself is not homogeneous anymore. However, it is covered by the frame flow, which itself can be viewed as a homogeneous flow on $SO(n,1)/\Gamma$ for some lattice $\Gamma \subset SO(n,1)$. Thus Mautner's and Moore's work proves the ergodicity of the frame and geodesic flow of any compact manifold of constant negative curvature. This justifies the third step in the proof of Mostow's rigidity theorem. The extensions of this to other symmetric spaces are similar in nature. In the next section we will derive Theorem 3.1 from asymptotic information about matrix coefficients.

3.2 Asymptotics of matrix coefficients

Let G be a connected semisimple Lie group. We will consider irreducible unitary representations π of G on a Hilbert space $\mathcal{H}$. Define the *matrix coefficient* or *correlation function* of v and $w \in \mathcal{H}$ as the function $\phi_{v,w} : G \to \mathbb{R}$ given by

$$g \to \langle \pi(g)v, w \rangle.$$

Harish-Chandra used matrix coefficients and their asymptotic properties as an essential tool in his seminal work on the representation theory of semisimple Lie groups (cf. e.g. [89]). His results were refined and extended by a number of people during the last two decades [185, 212, 99, 34, 98, 30, 144, 171]. While

I will describe the best asymptotic results further below, their proofs lie outside the scope of this article. Instead we will now discuss a vanishing result and its proof.

Let X be a locally compact topological space. We say that a function $f : X \to \mathbb{R}$ *vanishes at* ∞ if $f(x) \to 0$ as x leaves compact subsets of X. We also say that a sequence $x_n \in X$ *tends to* ∞ if x_n leaves any compact subset of X. The following result appeared in papers by T. Sherman, R. Howe and C. C. Moore, and R. J. Zimmer [185, 99, 212].

Theorem 3.2 *Let G be a connected semisimple real Lie group without compact factors. Suppose π is a unitary representation of G in a Hilbert space V for which no non-trivial normal subgroup G' has invariant vectors. Then the matrix coefficients of π vanish at ∞.*

Note that Theorem 3.1 follows immediately in case G is simple. Indeed, decompose $L^2(G/\Gamma)$ into unitary irreducible representations of G. Suppose some function f orthogonal to the constants is invariant under a homogeneous flow $\{a_t\}$. Then some component f' of f in some non-trivial irreducible subrepresentation π of $L^2(G/\Gamma)$ is non-zero and $\{a_t\}$-invariant. In particular, the matrix coefficient of f' does not vanish at ∞. Since G is simple and π is non-trivial, no non-trivial normal subgroup of G has an invariant vector. Thus Theorem 3.2 gives a contradiction. The general case in Theorem 3.1 requires some technicalities about irreducible lattices.

Next we will outline the proof of this decay result for the case $G = SL(2, \mathbb{R})$ (for a detailed treatment of this and the general case see [220]). The main idea of the proof is similar to that of Mautner's phenomenon. There, we used the commutation relation between geodesic and horospherical flow applied to the representation π itself. Here, we will exploit it in a more sophisticated way, by looking at the dual action of the geodesic flow on the set of characters for the horocyclic flow it defines.

Outline of proof : Recall that any element $g \in SL(2, \mathbb{R})$ can be written as a product $g = k\,a\,l$ where a is a diagonal matrix and k and l are orthogonal matrices. Clearly, a sequence g_n tends to ∞ if and only if the diagonal factor a_n in the above decomposition tends to ∞. Thus it suffices to prove that matrix coefficients vanish at infinity when restricted to the diagonal subgroup A.

Let N be the group of strictly upper triangular matrices. As N is isomorphic to $\mathbb{R}$, the irreducible unitary representations $\hat{N} = \hat{\mathbb{R}}$ of N are precisely the one-dimensional unitary representations π_θ given by multiplication by $e^{i\,\theta\,t}$ for $\theta \in \mathbb{R}$. Furthermore, any unitary representation of N decomposes as a direct integral of irreducible unitary representations. For $m \in \{\infty, 0, 1, 2, \ldots\}$, let $m\,\pi_\theta$ denote the direct sum of m copies of π_θ. We call m the *multiplicity* of π_θ. Thus, restricting π to N, there is a measure ν on $\hat{\mathbb{R}}$ and multiplicities $m_\theta \in \{\infty, 0, 1, 2, \ldots\}$ such that

$$\pi \mid_N = \int_{\hat{\mathbb{R}}} m_\theta \pi_\theta d\,\nu(\theta).$$

As A normalizes N, $a \in A$ also acts on the unitary dual $\hat{N}$ via $(a\,\pi_\theta)(t) = \pi_\theta(\,\mathrm{Ad}\,a^{-1}t)$. One calculates that $a\,\pi_\theta = \pi_{a^2\,\theta}$.

Now there is a dichotomy, either $\nu(0) = 0$ or $\nu(0) > 0$. In the first case, the support of $a^2\,\nu$ moves to infinity in $\hat{\mathbb{R}}$ as $a \to \infty$. This implies the vanishing of the matrix coefficients of π restricted to A. In the second case, when $\nu(0) = 0$, we get A- and N-invariant vectors in π. Note that this argument also shows that any A-invariant vector is N-invariant. By a similar argument, any A-invariant vector is fixed by the the strictly lower triangular subgroup N^-. Since A, N and N^- generate $SL(2, \mathbb{R})$, we conclude that either all matrix coefficients vanish at ∞ or there are $SL(2, \mathbb{R})$-invariant vectors. ◇

While the last theorem always insures the vanishing of the matrix coefficients, the rate of decay can be very slow in general. This can be seen as follows. As first shown by Fomin and Gelfand [62], the geodesic flow of a surface with constant negative curvature has countable Lebesgue spectrum (indeed, it is K and even Bernoulli). Thus there is an L^2-function f orthogonal to all its translates by g_T^n for some fixed T

and arbitrary n. Considering functions of the form $\sum_{i=0}^{\infty} \beta_i(g_{T\,i}\, f)$, we see that the decay can be as slow as the the decay of the l_2-norm of the tail of an l_2-sequence.

Harish-Chandra on the other hand found explicit exponential decay estimates for the matrix coefficients of sufficiently nice vectors [89]. These estimates were recently reproved and refined by M. Cowling, W. Casselman and D. Miličić and R. Howe [34, 30, 98] by various methods.

Recall that any locally compact group has maximal compact subgroups. For a Lie group, they are unique up to conjugacy. As an example, the group $SO(n)$ of orthogonal matrices of determinant 1 is a maximal compact subgroup for $SL(n, \mathbb{R})$.

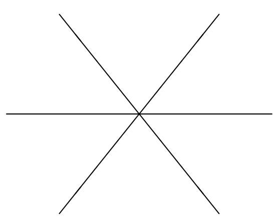
Let G be a connected semisimple Lie group with finite center as above. Fix a maximal compact subgroup K of G. A vector $v \in \mathcal{H}$ is called *K-finite* if the K-orbit of v spans a finite dimensional vector space. Let $\hat{K}$ denote the unitary dual of K. One can then decompose

$$\mathcal{H} = \oplus_{\mu \in \hat{K}} \mathcal{H}_\mu$$

where $\mathcal{H}_\mu$ is $\pi(K)$-invariant and the action of K on $\mathcal{H}_\mu$ is equivalent to $n\mu$ where n is an integer or $+\infty$, called the *multiplicity* of μ in $\mathcal{H}$. The K-finite vectors form a dense subset of $\mathcal{H}$. One calls $\mathcal{H}_\mu$ the *μ-isotypic component* of π.

Recall that a *Cartan subgroup* is a maximal abelian subgroup of G consisting of semisimple elements. It decomposes into a product of a torus and an $\mathbb{R}^k$ for some k. The $\mathbb{R}^k$-factor is called a *split Cartan subgroup* of G. We call it *maximal* if its dimension is maximal amongst all split Cartan subgroups.

Let A be a maximal split Cartan subgroup of G, and $\mathcal{A}$ its Lie algebra. For $G = SL(n, \mathbb{R})$ for example, we may take A to be the subgroup of diagonal matrices of determinant 1. Then $\mathcal{A}$ becomes the Lie subalgebra of trace 0 diagonal matrices. Recall that the *(restricted) roots* α are linear functions on $\mathcal{A}$ such that the eigenvalues of the adjoint representation of $x \in \mathcal{A}$ are given by the $\alpha(x)$ as α ranges over the roots. For $G = SL(n, \mathbb{R})$ and $\mathcal{A}$ the traceless diagonal matrices, the roots are just the functionals $e_i - e_j$ on $\mathcal{A}$, where e_i denotes the i-th diagonal entry of $x \in \mathcal{A}$.



In general, the kernels of the roots determine finitely many hyperplanes in $\mathcal{A}$. Removing these hyperplanes, we obtain finitely many connected components, called the *Weyl chambers*. Let us fix such a Weyl chamber $\mathcal{C}$, and call it the *positive* Weyl chamber. The roots whose hyperplanes bound $\mathcal{C}$ are a basis of $\mathcal{A}$. Call them the *elementary* roots. Any other root can be expressed as an integral linear combination of the elementary roots with either all coefficients nonnegative or nonpositive. Thus we can speak of the *positive* and *negative* roots. In fact, picking a basis of roots for $\mathcal{A}$ with this property, is equivalent to picking a positive Weyl chamber. In our basic example of $G = SL(n, \mathbb{R})$, we can choose $e_{i+1} - e_i$, $i = 2, \ldots, n$ as a set of elementary roots. For $n = 3$, the Weyl chambers are represented by the familiar picture on the left.

Call π *strongly L^p* if there is a dense subset of $\mathcal{H}$ such that for v, w in this subspace, $\phi_{v,w} \in L^p(G)$. Let ρ be half the sum of the positive roots on $\mathcal{A}$. R. Howe obtained the following estimate for matrix coefficients of K-finite vectors in 1980 [98, Corollary 7.2 and §7].

Theorem 3.3 *Let π be a strongly L^p-representation of G on $\mathcal{H}$. Let μ and ν be in $\hat{K}$. Then the matrix coefficients of $v \in \mathcal{H}_\mu$ and $w \in \mathcal{H}_\nu$ satisfy the estimate:*

$$| \phi_{v,w}(\exp tA) | \leq D \parallel v \parallel \parallel w \parallel \ \ dim\,\mu \ \ dim\,\nu \, e^{-\frac{t}{2p}\,\rho(A)}$$

where $A \in \bar{\mathcal{C}}$ and $D > 0$ is a universal constant.

Howe's version of the decay estimates is particularly strong, since it gives explicit and uniform control in terms of the norms of the K-finite vectors and some universal constants. The question remains which representations are strongly L^p. Fortunately, M. Cowling had already resolved it in 1979 [34].

Theorem 3.4 *Every irreducible unitary representation of G with discrete kernel is strongly L^p for some p. Furthermore, if $\mathcal{G}$ does not have factors isomorphic to $so(n,1)$ or $su(n,1)$ then p can be chosen independently of π.*

For applications in dynamics, it is important to obtain decay results for more general vectors, for example C^∞-functions on a manifold. These are easy corollaries of Howe's and Cowling's results. Here the uniformity in Howe's estimates becomes particularly important.

A vector $v \in \mathcal{H}$ is called C^∞ or *smooth* if the map $g \in G \to \pi(g)v$ is C^∞. Let $m = \dim K$ and $X_1, \ldots, X_m$ be an orthonormal basis of the Lie algebra $\mathcal{K}$ of K. Set $\Omega = 1 - \sum_{i=1}^m X_i^2$. Then Ω belongs to the center of the universal enveloping algebra of $\mathcal{K}$, and acts on the K-finite vectors in $\mathcal{H}$ since K-finite vectors are smooth. Now A. Katok and the author obtained the following estimate [114]:

Corollary 3.5 *Let v and w be C^∞-vectors in an irreducible unitary representation π of G with discrete kernel. Then there is a universal constant $E > 0$ and an integer $p > 0$ such that for all $A \in \bar{\mathcal{C}}$ and large enough l*

$$| \langle \exp(t\,A)v, w \rangle | \leq E\, e^{-\frac{t}{2p}\,\rho(A)} \parallel \Omega^l(v) \parallel \parallel \Omega^l(w) \parallel .$$

In fact, p can be any number for which π is strongly L^p. Furthermore, if $\mathcal{G}$ does not have factors isomorphic to $so(n,1)$ or $su(n,1)$, p only depends on G.

Note that v and w only need to be C^k with respect to K for some large k. Combining this with Moore's Theorem 3.1 as in [114], we obtain:

Corollary 3.6 *Let G be a semisimple connected Lie group with finite center. Let Γ be an irreducible cocompact lattice in G. Assume that $\mathcal{G}$ does not have factors isomorphic to $so(n,1)$ or $su(n,1)$. Let $f_1, f_2 \in L^2(G/\Gamma)$ be C^∞-functions orthogonal to the constants. Let $\mathcal{C}$ be a positive Weyl chamber in a maximal split Cartan $\mathcal{A}$. Then there is an integer $p > 0$ which only depends on G and a constant $E > 0$ such that for all $A \in \bar{\mathcal{C}}$ and all large l*

$$\langle (\exp tA)_*(f_1), f_2 \rangle \ \leq E\, e^{-\frac{t}{2p}\,\rho(A)} \parallel f_1 \parallel_l \parallel f_2 \parallel_l$$

Here $\parallel f \parallel_l$ denotes the l'th Sobolev norm of f.

This corollary extends to G locally isomorphic to $SO(n,1)$ or $SU(n,1)$ if we allow p to depend on the lattice Γ as well as on G. In fact, such a p is roughly inversely proportional to the bottom of the spectrum of the Laplacian on non-constant functions on the locally symmetric space $K \backslash G/\Gamma$. As $K \backslash G/\Gamma$ is compact, the latter is not 0. Thus one might hope for a positive resolution of the following question.

Problem 3.7 *Does the corollary extend to all G for a p that depends only on the lattice?*

Already for $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, this seems to be a very subtle problem.

Our results for the exponential decay of matrix coefficients for smooth functions extend to Hölder vectors and functions. This was first shown for representations of $SL(2,\mathbb{R})$ by M. Ratner [171]. C. C. Moore gave an alternative proof for arbitrary rank one groups in [144]. However, he had to assume that the vectors had a Hölder exponent bigger than $\dim K/2$ where K is the maximal compact subgroup. In a private communication, G. A. Margulis outlined an argument for arbitrary Hölder vectors for general G.

3.3 Higher rank hyperbolic abelian actions

The second application of the decay of matrix coefficients concerns the the rigidity of certain homogeneous actions of higher rank abelian groups. This is joint work of A. Katok and myself [114]. While our theorems hold for fairly general hyperbolic homogeneous actions, the so-called *standard partially hyperbolic actions*, I will restrict the outline of the proof here to the semisimple case. I already summarized our results for the Anosov case in Section 2.4.

Let me first introduce the notion of a partially hyperbolic action.

Definition 3.8 *Let A be $\mathbb{R}^k$ or $\mathbb{Z}^k$. Suppose A acts C^∞ and locally freely on a manifold M with a Riemannian norm $\| \, \|$. Call an element $g \in A$* partially hyperbolic *if there exist real numbers $\lambda > \mu > 0$, $C, C' > 0$ and a continuous splitting of the tangent bundle*

$$TM = E_g^+ + E_g^0 + E_g^-$$

such that for all $p \in M$, for all $v \in E_g^+(p)$ ($v \in E_g^-(p)$ respectively) and $n > 0$ ($n < 0$ respectively) we have for the differential $g_* : TM \to TM$

$$\| g_*^n(v) \| \leq C e^{-\lambda |n|} \| v \|$$

and for all $n \in \mathbb{Z}$ and $v \in E_g^0$ we have

$$\| g_*^n(v) \| \geq C' e^{-\mu |n|} \| v \| .$$

Furthermore, we assume that the distribution E_g^0 is uniquely integrable.

Call an A-action partially hyperbolic *if it contains a partially hyperbolic element. We also say that the A acts* normally hyperbolically *with respect to the foliation defined by E_g^0 We call E_g^+ and E_g^- its stable and unstable distribution respectively.*

Note that every Anosov flow and diffeomorphism is partially hyperbolic. More generally, we call a partially hyperbolic action *Anosov* if E_g^0 equals the tangent distribution of the orbit foliation of A for some partially hyperbolic element $g \in A$.

Our prime examples of such actions are algebraic. Let G be a connected semisimple Lie group of the non-compact type, A a maximal split Cartan and Γ a uniform lattice in G. Then the homogeneous action of A on G/Γ is partially hyperbolic. Let M be the compact part of the centralizer of A in G. Then the homogeneous action of A on G/Γ descends to an action of A on $M \setminus G/\Gamma$. The latter action is always an Anosov action, called the *Weyl chamber flow* on G/Γ. If we further assume that G does not have local factors isomorphic to $SO(n,1)$ or $SU(n,1)$, then all of these actions are standard partially hyperbolic actions.

Let us first describe how one reduces local rigidity of the standard Anosov actions to a cocycle theorem. By structural stability, any perturbation of an Anosov action is still Anosov, and C^0-conjugate to the original action α. By a geometric argument, one can see that the homeomorphism is a C^∞-diffeomorphism. Thus, up to a conjugacy, the perturbed action is a C^∞-*time change* of α, i.e. a C^∞-action whose orbits coincide with those of α. While this time change involves only small changes of time, we can actually exclude global time changes even for the standard partially hyperbolic $\mathbb{R}^k$-actions.

Theorem 3.9 [Katok-Spatzier, 1992] *All C^∞-time changes of a standard partially hyperbolic $\mathbb{R}^k$-action are C^∞-conjugate to the original action up to an automorphism.*

Every C^∞-time-change α^* of an action α determines a C^∞-cocycle $\beta : \mathbb{R}^k \times M \to \mathbb{R}^k$ via the equation

$$\alpha(a,x) = \alpha^*(\beta(a,x),x).$$

This is clear on orbits without isotropy. From this and the fact that most orbits of Anosov actions do not have isotropy, one can extend the cocycle everywhere.

If β is C^∞-cohomologous to a constant cocycle given by an automorphism $\phi : \mathbb{R}^k \to \mathbb{R}^k$, then α^* is C^∞-conjugate to $\alpha \circ \phi$. Thus it suffices to prove the following extension of Theorem 2.19.

Theorem 3.10 [Katok-Spatzier, 1992] *Any C^∞-cocycle $\beta : A \times M \to \mathbb{R}^m$ over a standard partially hyperbolic A-action is C^∞-cohomologous to a constant cocycle.*

I will now illustrate the proof of this theorem in the semisimple case. Thus let G be a connected semisimple Lie group of the non-compact type and real rank at least 2, A a maximal split Cartan and Γ a uniform lattice in G. Further assume that G does not have factors locally isomorphic to $SO(n,1)$ or $SU(n,1)$. Let A act on G/Γ by left translations. Pick a regular element $a \in A$, i.e. a does not lie on the wall of a Weyl chamber. I will show that β is cohomologous to the homomorphism $\rho(b) = \int_M \beta(b,x)dx$, or that $\beta - \rho$ is cohomologous to 0. Thus we may assume that β has 0 averages.

Let $f : G/\Gamma \to \mathbb{R}^k$ be the function given by $f(x) = \beta(a,x)$. Denote by $a\,f$ the function $a\,f(x) = f(ax)$. Define formal solutions of the cohomology equation by

$$P_a^+ = \sum_{k=0}^{\infty} a^k f \qquad \text{and} \qquad P_a^- = -\sum_{k=-\infty}^{-1} a^k f.$$

By our assumptions on G, we can apply Corollary 3.6 on the exponential decay of matrix coefficients of smooth functions on G/Γ. It implies that P_a^+ and P_a^- are not just formal solutions, but define distributions on G/Γ.

The key argument is to show that $P_a^+ = P_a^-$. Again this uses the exponential decay of matrix coefficients crucially. Since A has rank at least 2, we can pick $b \in A$ independent of a. Let $f^*(x) = \beta(b,x)$. Since β is a cocycle, it follows that

$$\sum_{k=-l}^{l} a^k bf - \sum_{k=-l}^{l} a^k f = \sum_{k=-l}^{l} a^{k+1} f^* - a^k f^* = a^{l+1} f^* - a_1^{-l} f^*.$$

Since the matrix coefficients decay, it follows that $P_a^+ - P_a^-$ is b-invariant. By exponential decay, the sum

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle a^k f, b^m g\rangle = \lim_{m\to\infty} 2m \sum_{k=-\infty}^{\infty} \langle a^k f, g\rangle$$

converges absolutely. It follows that $P_a^+ - P_a^- = 0$ as desired.

Note that P_a^+ is C^∞ along the stable manifold. For the first derivative for example, the derivatives $\frac{d}{dv}(f \circ a^k)$ decay exponentially for any vector v tangent to the stable manifold, and hence the sum $\sum_{k=0}^{\infty} \frac{d}{dv}(a^k \circ f)$ is absolutely convergent. Similarly, P_a^- is C^∞ along the unstable manifold. Since $P_a^+ = P_a^-$, we see that P_a^+ is C^∞ along both stable and unstable directions. The stable and unstable distribution together with their Lie brackets generate the whole tangent bundle. Using subelliptic estimates for sums of even powers of vectorfields in these directions, due to Rothschild, Nourrigat, Helffer and others, one can show that P_a^- is C^∞ on M. These estimates are very strong generalizations of the famed Hoermander square theorem. Let us note here that these subelliptic estimates themselves were developed using the harmonic analysis of nilpotent groups [178, 93].

While we used non-commutative harmonic analysis in the above, ordinary Fourier series arguments can be used to prove Theorem 3.9 for standard actions on tori. However, the superpolynomial decay of Fourier coefficients of smooth functions replaces the exponential decay estimates. Thus while the result depends on harmonic analysis in any of the cases, there is no uniform method of proof at this point of time.

3.4 Restrictions of representations to lattices and equivariant maps

M. Cowling and T. Steger used the fine decay estimates on matrix coefficients to see when restrictions of irreducible unitary representations of semisimple groups to lattices are irreducible and isomorphic [36]. This generalized earlier work of Steger for the case of $SL(2,\mathbb{R})$. For simplicity, I will state their theorem for irreducible lattices.

Let G be a semisimple group without compact factors. The irreducible subrepresentations of the left regular representation of G on $L^2(G)$ are called the *discrete series* representations of G. One can characterize the discrete series representations as the irreducible unitary representations with a matrix coefficient which belongs to $L^2(G)$.

Theorem 3.11 [Cowling-Steger, 1991] *Suppose Γ is an irreducible lattice in a connected semisimple group G with finite center and without compact factors. Suppose π and $\hat{\pi}$ are unitary representations of G. Then*

1. *if π is a discrete series representation of G, then $\pi\mid_\Gamma$ is reducible;*
2. *if π is not a discrete series representation, then $\pi\mid_\Gamma$ is irreducible;*
3. *if $\pi\mid_\Gamma$ and $\hat{\pi}\mid_\Gamma$ are irreducible and unitarily equivalent, then π and $\hat{\pi}$ are unitarily equivalent.*

A. Iozzi applied this result to actions of connected semisimple Lie groups G with finite center and without compact factors. She called an action on a space (X,μ), μ a finite G-invariant measure, *purely atomic* if the representation of G on $L^2(X,\mu)$ is a direct sum of irreducible representations [102].

Theorem 3.12 [Iozzi, 1992] *Let G be as above and Γ a lattice in G. Suppose G acts on X_1 and X_2 preserving finite invariant measures μ_1 and μ_2. Suppose the actions have purely atomic spectra and are either essentially free or essentially transitive. Then any measure preserving measurable Γ-equivariant map $\phi : X_1 \to X_2$ is G-equivariant.*

Iozzi first showed that the spectrum of such an action cannot be a sum of discrete series representations of G. Let K is a maximal compact subgroup of G. The decomposition of the discrete series representations into irreducible subrepresentations of K is given by the so-called *Blattner formula*, which was established by H. Hecht and W. Schmid [182, 92]. In particular, one sees that discrete series representations do not contain K-invariant vectors. Suppose now that the spectrum of the G-action on X_i consists only of discrete series representations. Then there are no K-invariant non-constant functions on the X_i. Thus K acts transitively, and G cannot act with discrete stabilizer. Since ϕ is measure preserving, ϕ gives rise to a unitary intertwining operator T of the restrictions of the representations to the lattice. By Theorem 3.11, T intertwines the non-discrete series parts $\mathcal{H}_0^i$ of the G-representations. Now consider the factor Y_i of X_i with Borel algebra generated by the $\mathcal{H}_0^i$. Then the Y_i have G-actions. Since T intertwines the G-actions, there is a G-equivariant map $Y_1 \to Y_2$. To complete the proof, Iozzi showed that K acts transitively on the fibers of the X_i over the Y_i. This uses the pure atomicity of the spectrum.

4 Amenability

4.1 Definitions and basic results

Let us begin with the representation theoretic definition of amenability. Call an action of a (locally compact topological) group G on a compact convex subset W of a locally convex topological vector space *affine* if for all $w_1, w_2 \in W$, all $0 \le t \le 1$ and all $g \in G$, we have $g\,(t\,w_1 + (1-t)\,w_2) = t\,(g\,w_1) + (1-t)\,(g\,w_2)$. Call G *amenable* if every continuous affine action of G on a compact convex subset W of a locally convex topological vector space fixes some $w \in W$, i.e. $g\,w = w$ for all $g \in G$. In fact it is sufficient to require such fixed points for affine actions on subsets of duals of separable Banach spaces W.

There are several other conditions equivalent to amenability such as the existence of an invariant mean on G or the existence of an invariant probability measure for any action of G on a compact non-empty topological space X. A direct characterization of amenability in terms of the group can be given using Følner sets [75]. There are also other representation theoretic criteria. The *regular representation* of G is the representation of G on $L^2_\mu(G)$, where μ is Haar measure. A unitary representation π is called *weakly contained* in a unitary representation σ if every matrix coefficient of π is a uniform limit on compact subsets of matrix coefficients of σ. Then amenability is equivalent to the trivial one-dimensional representation (and in fact any irreducible unitary representation) being weakly contained in the regular representation of G [75]. Various people have also considered an extension of amenability, K-amenability, a property of the K-theory of the group (cf. e.g. [105]). While every amenable group is K-amenable there are also certain rank 1 groups such as $SL(2,\mathbb{R})$ and $SU(1,1)$ that are K-amenable. The dynamic impact of this property has not been studied.

Any abelian or solvable group is amenable. Conversely, amenable connected Lie groups are always compact extensions of solvable Lie groups [72]. On the other hand, there are very complicated amenable discrete groups.

The representation theoretic notion of amenability as a fixed point property quite easily generalizes to group actions. This was first done by R. J. Zimmer in 1978 [213]. Recall the notion of a measurable cocycle from Section 2.4, before Theorem 2.8. Suppose we are given an action of G on X with quasi-invariant measure ν, a measurable cocycle $\beta : G \times X \to H$ and an action of H on a measure space Y. Then a measurable map $s : X \to Y$ is called a *section* of α if for ν-a.e. $x \in X$ and all $g \in G$ we have $s(g\,x) = \beta(g,x)\,s(x)$. Let E be a separable Banach space. Consider cocycles β taking values in the isometry group $Iso(E)$. Let β^* denote the dual cocycle taking values in $Iso(E^*)$. Let E^*_1 denote the unit ball of E^*. A *(Borel) field of convex convex sets* A_x, $x \in X$, is a family of convex compact subsets of E^*_1 such that $\{x, e) \mid e \in A_x\}$ is Borel. Call such a field A_x *β-invariant* if $A_{g\,x} = \beta(g,x)^*(A_x)$.

Definition 4.1 *An action α of G on a measure space (X,ν) is* amenable *if for any separable Banach space E, every cocycle $\beta : G \times X \to Iso(E)$ and every β-invariant field of convex sets A_x, there is a section s of β^* such that $s(x) \in A_x$ for ν-a.e. $x \in X$.*

It is easy to rephrase this condition as an ordinary fixed point condition in terms of the representation of G on $L^1(X,E)$ skewed by β [220, Section 4.3].

Any action of an amenable group is amenable. Furthermore, a homogeneous action of G on G/H, H a closed subgroup of G, is amenable if and only if H is amenable [220]. This gives us many examples of amenable actions. The next example lies at the heart of many applications of amenability to rigidity theory.

Example 4.2 Let Γ be a lattice in $SL(2,\mathbb{R})$. The action of Γ on the sphere at infinity S^1 of $\mathcal{H}^2$ is equivalent to the action of Γ on $SL(2,\mathbb{R})/H$ where H is the subgroup of upper triangular matrices. For any closed subgroup L of $SL(2,\mathbb{R})$, the action of Γ on $SL(2,\mathbb{R})/L$ is amenable if and only if the action of L on $SL(2,\mathbb{R})/\Gamma$ is amenable. Thus the action of Γ on S^1 is amenable (w.r.t. Lebesgue measure on S^1) since H is solvable and hence amenable.

More generally, R. J. Zimmer and the author showed in 1991 that the action of any discrete group of isometries on the sphere at infinity of a complete simply-connected Riemannian manifold with sectional curvature $-b^2 \leq K \leq -a^2 < 0$ is amenable with respect to any quasi-invariant measure [187, 189]. One says that the action is *universally amenable*. This was further generalized to actions of Gromov-hyperbolic groups on their spheres at infinity by S. Adams [1].

While the above shows the amenability of the action of a lattice on the sphere at infinity of a globally symmetric space of negative curvature, the situation is somewhat different for a semisimple group G of the noncompact type of real rank at least 2. For such G, it is more suitable to consider the so-called *Furstenberg boundaries*. These are the homogeneous spaces G/P where P is a parabolic in G (i.e. an algebraic subgroup P such that G/P is compact). There are 2^k such parabolics up to conjugacy where k is the real rank of G (including $P = G$). These boundaries actually appear as orbits of the G-action on the sphere at infinity of the globally symmetric space G/K, K a maximal compact subgroup of G. Up to conjugacy, there is exactly one parabolic of smallest dimension, the so-called *minimal parabolic*. It is always a compact extension of a solvable group. Hence we see as above that the action of a lattice on the maximal boundary G/P, P a minimal parabolic, is amenable.

4.2 Amenability, superrigidity and other applications

The following lemma lies at the heart of most of the applications of amenability to rigidity. For the special case of G-actions on maximal boundaries, it is due to Furstenberg [69]. Denote by $\mathcal{P}(Y)$ the set of probability measures on a space Y.

Lemma 4.3 [Furstenberg, 1973, Zimmer, 1980] *Let G act amenably on a space X with quasi-invariant measure μ. Suppose $\beta : G \times X \to H$ is a cocycle into a group H. If H acts on a compact metric space Y, then there exists a β-invariant section $X \to \mathcal{P}(Y)$.*

The lemma readily follows from the definition of amenability, applied to the Banach space of continuous functions on Y.

As a first application, let me now explain the first step in the proof of Margulis' superrigidity theorem. Let Γ be a lattice in a semisimple group G without compact factors and of rank at least 2. Suppose $\psi : \Gamma \to H$ is a homomorphism into a connected Lie group H. For simplicity, we will assume that H is also semisimple without compact factors. As in Mostow's proof of strong rigidity, the first goal is to find a Γ-equivariant map ϕ between boundaries of G and H. Margulis originally used the multiplicative ergodic theorem to find such a map [129]. In his extension of Margulis' superrigidity theorem to cocycles, Zimmer later modified Margulis' approach. He used only the amenability of the Γ-action on the maximal boundary G/P of G to construct ϕ [216]. In fact, let B be a maximal boundary of H. By Lemma 4.3 there is a β^*-invariant section $s : G/P \to \mathcal{P}(B)$ with values in the space of probability measures $\mathcal{P}(B)$. Zimmer then showed, using a method of Furstenberg, that the action of H on $\mathcal{P}(B)$ is smooth, i.e. all orbits are locally closed [65, 70, 214]. Let $\pi : \mathcal{P}(B) \to \mathcal{P}(B)/H$ be the projection. Then the composition $\pi \circ s : G/P \to \mathcal{P}(B)/H$ is a Γ-invariant map into a standard probability space. Note that P acts ergodically on G/Γ by the Mautner-Moore phenomenon. Hence Γ acts ergodically on G/P, and $\pi \circ s$ is a.e. constant. This means that s essentially takes values in a single orbit H/L of H on $\mathcal{P}(B)$. One can show that L is a parabolic. Therefore we get the desired measurable Γ-equivariant mapping. Note that we get such a map even in real rank one. However, to show that this map is an algebraic map $G/P \to H/L$, Margulis used the higher rank condition. To get a homomorphism $G \to H$ then is easy.

There are quite a few other applications of this idea. To start with, Zimmer proved his superrigidity theorem for cocycles along the same lines [216]. He later used it, in conjunction with an extension of Margulis' theorem about equivariant measurable quotients of boundaries, to prove the following theorem about *Riemannian foliations* [218]. These are foliations of a measure space (X, ν) such that a.e. leaf has a Riemannian structure. Note that the notions of cocycle and thus amenability naturally extend to equivalence relations, and in particular to foliations.

Theorem 4.4 [Zimmer, 1982] *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Riemannian ergodic foliations with transversely invariant measure and finite total volume. Suppose that a.e. leaf of $\mathcal{F}_1$ is simply-connected and complete, and that the sectional curvatures are negative and uniformly bounded away from 0. Further suppose that $\mathcal{F}_2$ is irreducible and that a.e. leaf of $\mathcal{F}_2$ is isometric to a symmetric space of the noncompact type of rank at least 2. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are not transversely equivalent. Furthermore, $\mathcal{F}_1$ and $\mathcal{F}_2$ are not amenable, and thus not transversely equivalent to the orbit foliation of an action of an amenable group.*

The idea of an amenable foliation had already proved useful when Zimmer extended the Gromoll-Wolf and Lawson-Yau result on solvable fundamental groups of manifolds of non-positive curvature to foliations [219]. The general philosophy is that assumptions on the fundamental group of a manifold can be replaced by suitable hypotheses about foliations.

Theorem 4.5 [Zimmer, 1982] *Let $\mathcal{F}$ be an amenable Riemannian measurable foliation with transversely invariant measure and finite total volume. Suppose a.e. leaf is simply-connected and complete, and has non-positive sectional curvature. Then a.e. leaf is flat.*

Again one main idea is to find a section of probability measures on the spheres at infinity of the leaves invariant under the "holonomy" of $\mathcal{F}$.

Zimmer and I obtained restrictions on the fundamental group of spaces on which a higher rank semisimple group can act [189].

Theorem 4.6 [Spatzier-Zimmer, 1991] *Let G be a connected simple Lie group with finite center, finite fundamental group, and real rank at least 2. Suppose G acts on a closed manifold M preserving a real analytic connection and a finite measure. Then $\pi_1(M)$ cannot be isomorphic to the fundamental group Λ of a complete Riemannian manifold N with sectional curvature $-a^2 \leq K \leq -b^2 < 0$ for some real numbers a and b.*

S. Adams generalized this theorem to Gromov hyperbolic groups and spaces [1, 3]. The proof of Theorem 4.6 is is inspired by that of Theorem 4.4. However, we use the amenability of the boundary actions twice, in very different ways.

Outline of Proof: For simplicity, assume G is simply-connected. Suppose $\pi_1(M)$ is isomorphic to some Λ as in Theorem 4.6. View the universal cover $\tilde{M}$ as a Λ-principal bundle over M. Then the lift of the G-action to $\tilde{M}$ acts by bundle automorphisms, and gives rise to a cocycle $\beta : G \times M \to \Lambda$. Let P be a minimal parabolic of G. Then the action of G on $M \times G/P$ is amenable. Let $\tilde{\beta}$ be the lift of β to $M \times G/P$. By Lemma 4.3, there is $\tilde{\beta}$-invariant section $\phi : M \times G/P \to \mathcal{P}(N_\infty)$, where N_∞ is the sphere at infinity of N, and $\mathcal{P}(N_\infty)$ is the set of probability measures on N_∞. Let $\psi : M \times G/P \to M \times \mathcal{P}(N_\infty)$ be the map $\psi(m, x) = (m, \phi(m, x))$. Then project the product measure class on $M \times G/P$ to $M \times \mathcal{P}(N_\infty)$. Since we can project to the first factor of $M \times \mathcal{P}(N_\infty)$, we see that $M \times \mathcal{P}(N_\infty)$ lies in between $M \times G/P$ and M. Zimmer's generalization of Margulis' measurable quotient theorem asserts that $M \times \mathcal{P}(N_\infty)$ is of the form $M \times G/P^*$ for some parabolic P^* of G.

Under the geometric assumptions of Theorem 4.6, M. Gromov showed that the G-action on $\tilde{M}$ is proper on a set of full measure. Hence β does not take values in a finite subgroup of Λ. In fact, no restriction of β to a noncompact closed subgroup of G is cohomologous to a cocycle taking values in a finite subgroup.

On the other hand, the section $\phi : M \times G/P \to \mathcal{P}(N_\infty)$ gives rise to a $\beta \mid_{P \times M}$-invariant map $\phi_0 : M \to \mathcal{P}(N_\infty)$. A probability measure on $\mathcal{P}(N_\infty)$ invariant under an infinite group of isometries is necessarily atomic with one or two atoms, due to the negative curvature on N. As a cocycle analogue, ϕ_0 as above is supported on one or two points for a.e. $m \in M$ unless $\beta \mid_{P \times M}$ is cohomologous to a cocycle taking values in a finite subgroup of Λ. The latter is not possible by the last paragraph. Let $Q = (N_\infty \times N_\infty)/S_2$ where S_2 is the permutation group in two letters. By the above, $M \times \mathcal{P}(N_\infty) = M \times G/P^*$ is isomorphic to the skew product $M \times_\beta Q$.

If $P^* \neq G$ there is a non-compact closed abelian subgroup $A \subset P^*$ that fixes a non-atomic probability measure μ on G/P^*. Under the above isomorphism, μ corresponds to a $\beta\mid_{A\times M}$-invariant map $\theta : M \to \mathcal{P}(Q)$. Lifting θ to a map into $\mathcal{P}(N_\infty \times N_\infty)$ we see that the essential range of θ is supported on atomic measures on Q. This contradicts the fact that μ is non-atomic.

We conclude that $P^* = G$. This implies that there is a β-invariant map $F : M \to \mathcal{P}(N_\infty)$. Again the essential range of F lies in $Q = (N_\infty \times N_\infty)/S_2$. Since N is negatively curved, N_∞ is universally amenable, and so is Q (cf. Example 4.2). Endowing Q with the image of the measure on M, we get a map from the G-action on M into an amenable action. Since G is Kazhdan (cf. Section 5), almost the opposite of amenability, this leads to a contradiction, unless β is cohomologous to a cocycle into a finite subgroup. As above, the latter is impossible by Gromov's result. ◇

Amenability of the boundary action and Lemma 4.3 were used in a novel way in

Theorem 4.7 [Zimmer, 1983] *Let Γ be a lattice in a connected simple Lie group with finite center or a fundamental group of a closed manifold of negative curvature. Suppose Γ acts essentially freely and ergodically on S, preserving a finite measure. If this action is orbit equivalent to a product action of two discrete groups Γ_1 and Γ_2 on $S_1 \times S_2$, then either S_1 or S_2 is essentially finite.*

S. Adams generalized this to Gromov hyperbolic groups in [2]. Let us illustrate the idea of the proof very roughly in case Γ is the fundamental group of a manifold M of negative curvature. First pick amenable subrelations of the actions of Γ_i on S_i. By Lemma 4.3, we get sections $S \to \mathcal{P}(M_\infty)$, invariant under the subrelations. Since the subrelations commute with the full action on the other factor, one can conclude that these sections are actually invariant under the full Γ-action. Thus again, this section takes values in atomic measures and hence the Γ-action is an extension of an amenable action, and hence amenable. Since Γ preserves a finite measure, this implies that Γ is amenable. This is impossible.

5 Kazhdan's Property

5.1 Definitions and basic results

Rigidity properties typically are strongest for higher rank semisimple Lie groups and fail for $SL(2,\mathbb{R})$. For the other rank one groups, they may or may not fail. For example, Mostow's strong rigidity theorem holds for all of them. Superrigidity and arithmeticity are only known to hold for $Sp(n,1)$ and F_4^{-20}. In this section we will introduce a representation theoretic property, called Kazhdan's property (T), which falls right in between higher rank and just excluding $SL(2,\mathbb{R})$. D. Kazhdan discovered it in 1967. He realized that for $SL(3,\mathbb{R})$, the trivial representation is isolated within the unitary representations [119]. For $SL(2,\mathbb{R})$ on the other hand, it is well known that the trivial representation is not isolated. He used this property to show that lattices in higher rank simple Lie groups of the non-compact type are finitely generated, and their first Betti number vanishes. Kazhdan's property (T) has proven amazingly successful in the study of the dynamics of semisimple groups. Interestingly, its range of validity within semisimple groups coincides exactly with that of of the superrigidity and arithmeticity theorems. I will now introduce this property in detail, and discuss a handful of representative applications to rigidity theory. I refer to the survey of P. de la Harpe and A. Valette for a more complete discussion of this property [90]. They also discuss some other exciting applications, such as telephone networks, that lie outside our presentation.

Let G be a locally compact second countable group. Let π be a unitary representation of G on a Hilbert space V. For any $\varepsilon > 0$ and compact subset K of G, we call a unit vector $v \in V$ (ε, K)-*invariant* if $\| \pi(k)\, v - v \| < \varepsilon$ for all $k \in K$.

Definition 5.1 [Kazhdan, 1967] *Call G a* Kazhdan group *if any unitary representation of G which has (ε, K)-invariant vectors for all compact subsets K and $\varepsilon > 0$, has G-invariant vectors.*

Any compact group is Kazhdan, as follows easily from a standard averaging argument. On the other hand, amenable Kazhdan groups are necessarily compact. Any connected semisimple real Lie group is Kazhdan provided that it does not have $SO(n, 1)$ or $SU(n, 1)$ as a local factor. This was shown by D. Kazhdan for the higher rank groups and by B. Kostant for the rank one groups [119, 120]. P. de la Harpe and A. Valette gave a streamlined proof for the rank one groups in [90]. In the p-adic case, only the higher rank or compact semisimple groups are Kazhdan [90]. S. P. Wang showed in 1975 that certain skew products such as $SL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ are Kazhdan. To get examples of discrete groups, D. Kazhdan observed that a lattice in a group G is Kazhdan if and only if G is Kazhdan. Finally note that any quotient group of a Kazhdan group is Kazhdan.

Let me note here that all known examples of Kazhdan groups are obtained via the above procedures, thus eventually through the (more or less) explicit representation theory of a Lie group. The geometry of Kazhdan groups is not well understood. E. Ghys for example asked:

Problem 5.2 [Quasi-isometric Invariance] *Suppose two finitely generated groups Γ_1 and Γ_2 are quasi-isometric with respect to their word metrics. Suppose Γ_1 is Kazhdan. Is Γ_2 necessarily Kazhdan?*

Note that the negatively curved locally symmetric spaces with Kazhdan fundamental groups all achieve sectional curvatures -1 and -4. Little is known for variable curvature.

Problem 5.3 [Pinching] *Let M be a closed Riemannian manifold with sectional curvatures $-4 < K < -1$. Can $\pi_1(M)$ be Kazhdan?*

Fundamental groups of closed manifolds with sectional curvature very close to -1, diameter bounded above and volume bounded below, are never Kazhdan. This is an immediate consequence of Gromov's compactness theorem for the space of geometrically bounded Riemannian metrics.

D. Kazhdan's original motivation for considering Kazhdan groups lay in the following result.

Theorem 5.4 [Kazhdan, 1967] *Let Γ be a discrete Kazhdan group. Then Γ is finitely generated and its abelianization is finite.*

Proof: List the elements of Γ by $\gamma_1, \ldots, \gamma_n, \ldots$. Let Γ_n denote the subgroup generated by $\gamma_1 \ldots \gamma_n$. Set $\pi = \oplus L^2(\Gamma/\Gamma_n)$. Since for any n, $\gamma_1 \ldots \gamma_n$ fix a vector, π has almost invariant vectors for any ε and finite subset of Γ. Since Γ is Kazhdan, there is a fixed unit vector v in π. Then the component v_n of v in any $L^2(\Gamma/\Gamma_n)$ is Γ-invariant. Thus for some n, v_n is a non-zero constant L^2-function on Γ/Γ_n. This implies that Γ/Γ_n is finite, and hence that Γ is finitely generated.

Since Γ is finitely generated, so is its abelianization $\Gamma/[\Gamma, \Gamma]$. It $\Gamma/[\Gamma, \Gamma]$ is infinite, it factors through $\mathbb{Z}$. As quotient groups of Kazhdan groups are Kazhdan, this is a contradiction. $\diamond$

I will now discuss some of the many applications of Kazhdan's property to rigidity theory.

5.2 Lorentz actions

In 1984, R. J. Zimmer obtained the following remarkable result about actions of Kazhdan groups on Lorentz manifolds [221]. His analysis rides on an understanding of the continuous homomorphisms from

Kazhdan groups into non-Kazhdan Lie groups.

Lemma 5.5 *Let Γ be Kazhdan, G a connected simple Lie group which is not Kazhdan. Then the image of any homomorphism $\rho : \Gamma \to G$ is contained in a compact subgroup of G.*

Proof: Let π be a unitary representation of G which does not contain a G-fixed vector. By Theorem 3.2 on the vanishing of matrix coefficients for π and G, we conclude that $\rho(\Gamma)$ is precompact. $\diamond$

The lemma actually extends to real algebraic groups G with all simple factors locally isomorphic to $SO(n,1)$ and $SU(n,1)$ [221, Corollary 20]. The cocycle analogue of the last lemma holds true as well [183, 217, 221]. The proof uses two more ingredients from harmonic analysis. For one, unitary representations of G as above have locally closed orbits provided that the π-image of the projective kernel of π is closed. Also, simple groups with faithful finite-dimensional unitary representations are compact.

Theorem 5.6 [Zimmer, 1984] *Let a Kazhdan group Γ act ergodically on a standard probability space S, preserving a measure μ. Let H be a real algebraic group. Then any cocycle $\beta : S \times \Gamma \to H$ is cohomologous to a cocycle taking values in an algebraic Kazhdan subgroup of H.*

For amenable ranges, this is due to K. Schmidt and R. J. Zimmer [183, 217].

Now suppose Γ acts on a compact Lorentz manifold preserving the Lorentz structure. Then the derivative cocycle takes values in $O(n,1)$. By Theorem 5.6, there is a measurable framing of the tangent bundle such that the derivative cocycle takes values in a compact group. Averaging a Riemannian structure for each such compact group, we see that there is a measurable Riemannian metric invariant under Γ. Using higher order jet bundles and Sobolev theory, Zimmer improved this conclusion as follows [221, 222].

Theorem 5.7 [Zimmer, 1984] *Let a Kazhdan group Γ act isometrically on a closed Lorentz manifold M. Then Γ preserves a C^∞-Riemannian metric on M. Thus the action is C^∞-equivalent to an action of Γ on a homogeneous space K/K_0 of a compact group K via a homomorphism $\Gamma \to K$.*

Lie group actions by Lorentz transformations are even more restricted. Zimmer showed that then either the Lie group is locally isomorphic to a product of $SL(2,\mathbb{R})$ and a compact group or it is amenable with nilradical at most of step 2 [224]. Lorentz actions have been further analyzed by G. D'Ambra and M. Gromov, using geometric means [78, 40].

5.3 Infinitesimal and local rigidity of actions

Recall A. Weil's local rigidity result. Namely, any small deformation of a uniform lattice Γ in a semisimple group G is conjugate to the lattice, provided that G does not have $SL(2,\mathbb{R})$ as a local factor [205, 206]. He actually showed "infinitesimal rigidity", namely that the cohomology $H^1(\Gamma, \mathrm{Ad}\,)$ vanishes. Applying the implicit function theorem, he then deduced local rigidity from that.

This motivates the following definition of infinitesimal rigidity for group actions. Suppose a group Γ act smoothly on a manifold M. Denote by $\mathcal{V}\rceil\rfloor\sqcup(M)$ the space of smooth vector fields on M. Call the Γ-action *infinitesimally rigid* if the first cohomology $H^1(\Gamma, \mathcal{V}\rceil\rfloor\sqcup(M))$ vanishes. Due to the delicate nature of the implicit function theorem in infinite dimensions, there is no clear connection between infinitesimal and local rigidity.

Let G be a connected semisimple Lie group G with finite center and without compact factors, and Γ a uniform lattice in G. Given a homomorphism $\pi : G \to H$ of G into another Lie group H and a uniform lattice $\Lambda \subset H$, then Γ acts naturally on H/Λ. R. J. Zimmer proved infinitesimal rigidity for such actions in [227].

Theorem 5.8 [Zimmer, 1990] *Assume that G does not have factors locally isomorphic to $SO(n,1)$ or $SU(n,1)$ and that $\pi(\Gamma)$ is dense in H. Then the Γ-action on H/Λ is infinitesimally rigid.*

Let me indicate how Kazhdan's property enters the proof. Let $\mathcal{V}\rceil\rfloor\sqcup_2(M)$ denote the space of L^2-vector fields on M. Zimmer first showed that the action is L^2-*infinitesimally rigid*, i.e. the canonical map

$$H^1(\Gamma, \mathcal{V}\rceil\rfloor\sqcup(M)) \to H^1(\Gamma, \mathcal{V}\rceil\rfloor\sqcup_2(M))$$

is zero. Thus there always is an L^2-coboundary, and Theorem 5.8 becomes a regularity theorem. To show L^2-*infinitesimal rigidity*, note that for $M = H/\Lambda$, the Γ-action on $\mathcal{V}\rceil\rfloor\sqcup(M)$ is isomorphic with the Γ-action on the space of functions from M to the Lie algebra $\mathfrak{H}$ of H where Γ acts via $\mathrm{Ad}\circ\pi$. Generalizing the calculations of Y. Matsushima and S. Murakami, Zimmer showed L^2-*infinitesimal rigidity* for all the non-trivial irreducible components of $\mathrm{Ad}\circ\pi$ [136, 227]. For the trivial representations contained in $\mathrm{Ad}\circ\pi$, the first cohomology in the L^2-functions already vanishes. The latter follows from the following characterization of Kazhdan's property due to J. P. Serre (in a letter to A. Guichardet) [90].

Theorem 5.9 [Serre] *A locally compact group G is Kazhdan if and only if for all unitary representations ρ of G, $H^1(G, \rho) = 0$.*

Note that these arguments apply to other finite dimensional representations besides $\mathrm{Ad}\circ\pi$.

In his thesis in 1989, J. Lewis managed to adapt Zimmer's arguments to certain actions of non-uniform lattices [121].

Theorem 5.10 [Lewis, 1989] *The action of $SL(n,\mathbb{Z})$ on the n-torus T^n is infinitesimally rigid for all $n \geq 7$.*

Deformation rigidity for this and other toral actions of lattices was obtained by S. Hurder [101]. A. Katok, J. Lewis and R. J. Zimmer later obtained local and semi-global rigidity results [110, 111, 112].

Theorem 5.11 [Hurder, Katok, Lewis, Zimmer, 1990's] *The standard action of $SL(n,\mathbb{Z})$ or any subgroup of finite index on the n-torus is locally C^∞-rigid.*

The Kazhdan property of $SL(n,\mathbb{Z})$ is used here to see that any small perturbation of the action preserves an absolutely continuous probability measure. N. Qian recently announced the deformation rigidity for "most" actions of irreducible higher rank lattices Γ on tori by automorphisms [161, 162, 163, 164].

R. J. Zimmer also obtained a local rigidity result for isometric actions of Kazhdan groups in [225].

Theorem 5.12 [Zimmer, 1987] *Let a Kazhdan group Γ act on a closed manifold preserving a smooth Riemannian metric. Then any small enough volume preserving ergodic perturbation of the action leaves a smooth Riemannian metric invariant.*

5.4 Discrete spectrum

Let a locally compact group G act on a measure space X preserving a probability measure μ. We say that the action has *discrete spectrum* if $L^2(X,\mu)$ decomposes into a direct sum of finite dimensional representations. Furthermore call an action of G *measurably isometric* if it is measurably conjugate to an action of G on a homogeneous space K/K_0 of a compact group K via a homomorphism $G \to K$. Measurably isometric actions always have discrete spectrum by the Peter-Weyl theorem. J. von Neumann in the commutative case and G. W. Mackey in general proved the converse [204, 125].

Note that measurably isometric actions always leave a measurable Riemannian metric invariant. The converse however is not true in general. A. Katok constructed an example of a volume preserving weak mixing diffeomorphism of a closed manifold which preserves a measurable Riemannian metric. For Kazhdan groups on the other hand, R. J. Zimmer obtained the complete equivalence in [228].

Theorem 5.13 [Zimmer, 1991] *Let a discrete Kazhdan group Γ act smoothly on a closed manifold preserving a smooth volume. Then the action has discrete spectrum if and only if Γ preserves a measurable Riemannian metric. Furthermore, if Γ is an irreducible lattice in a higher rank semisimple group, then discrete spectrum is also equivalent to the vanishing of the metric entropy for every element $\gamma \in \Gamma$.*

It is fairly easy to see that the Γ-action has some discrete spectrum. For every point $m \in M$, approximate the Γ-invariant measurable metric by a smooth metric ω_m in a neighborhood of m. Let $f_{r,m}$ be the normalized characteristic function of a ball of size r for ω_m about m. Set $F_r(m,x) = f_{r,m}(x)$ for $(x,m) \in M \times M$. For any finite set K in Γ and $\varepsilon > 0$, there is a small enough r such that F_r is (ε, K)-invariant. This follows since the original measurable metric is Γ-invariant. Since Γ is Kazhdan, F_r is close to a Γ-invariant function in $L^2(M \times M)$. This gives rise to a non-trivial Γ-invariant finite dimensional subspace of $L^2(M)$.

5.5 Ruziewicz' problem

By uniqueness of Haar measure, there is a unique countably additive rotation-invariant measure on any sphere S^n. S. Ruziewicz asked if the finitely additive rotation-invariant measures, defined on all Lebesgue measurable sets, are also unique. For the circle, S. Banach found other such measures in 1923. The problem remained open for the higher dimensional spheres until the early 80's. Then G. A. Margulis and D. Sullivan showed independently, using a partial result of J. Rosenblatt, that such measures do not exist on S^n for $n > 3$ [132, 194]. Their main idea is that $SO(n+1)$ for $n > 3$ contains discrete Kazhdan groups Γ dense in $SO(n+1)$. J. Rosenblatt on the other hand showed that a finitely additive rotation-invariant measure distinct from Lebesgue measure μ gives rise to Γ-almost invariant vectors in the orthogonal complement to the constants in $L^2(S^n, \mu)$. By Kazhdan's property, there are Γ- and hence $SO(n+1)$-invariant non-constant functions, a contradiction. The remaining cases, S^2 and S^3, were resolved by V. G. Drinfel'd in 1984 [46]. Though $SO(3)$ and $SO(4)$ do not contain discrete Kazhdan groups, he was able to exhibit discrete subgroups for which certain unitary representations do not contain almost invariant vectors. Drinfel'd's approach used deep theorems in number theory. To summarize, we have the complete resolution of Ruziewicz' problem.

Theorem 5.14 [Rosenblatt,Margulis,Sullivan,Drinfeld, 1980's] *For $n > 1$, there is a unique rotation invariant finitely additive measure on S^n, defined on all Lebesgue measurable sets.*

G. A. Margulis also resolved the analogous problem for Euclidean spaces [132]. There we have uniqueness on $\mathbb{R}^n$ exactly when $n > 2$. K. Schmidt has further analyzed the connection between actions of Kazhdan groups and unique invariant means [183]. I refer to [90] for a detailed exposition of the Ruziewicz' problem as well as other interesting applications of Kazhdan's property outside rigidity theory.

5.6 Gaps in the Hausdorff dimension of limit sets

Let $\mathcal{H}^n$ denote either a quaternionic hyperbolic space or the Cayley plane. Compactify $\mathcal{H}^n$ by a sphere S as usual. The *limit set* $L(\Gamma)$ of a discrete group of isometries Γ of $\mathcal{H}^n$ is the set of accumulation points of a Γ-orbit of a point $x \in \mathcal{H}^n$ in S. This is independent of the choice of initial point x. The *ordinary set* $O(\Gamma)$ is the complement $S \setminus L(\Gamma)$ of $L(\Gamma)$ in S. Then Γ acts properly discontinuously on $\mathcal{H}^n \cup O(\Gamma)$. Call Γ *geometrically cocompact* if $\mathcal{H}^n \cup O(\Gamma)/\Gamma$ is compact. This is a generalization of convex cocompact groups on real hyperbolic space. K. Corlette used Kazhdan's property to estimate a gap in the Hausdorff dimension of the limit set of such groups in [31].

Theorem 5.15 [Corlette, 1990] *A geometrically cocompact discrete subgroup of $Sp(n,1)$ (or F_4^{-20}) is either a lattice or its limit set has Hausdorff codimension at least 2 (respectively 6). Furthermore, $\mathcal{H}^n/\Gamma$ has at most one end.*

Corlette proved similar results for higher rank semisimple groups. However, it is not clear if there are any non-trivial examples of geometrically cocompact groups in higher rank.

To outline the proof of Theorem 5.15, Corlette first established a connection between the bottom of the spectrum λ_0 of the Laplacian on $\mathcal{H}^n/\Gamma$ and the Hausdorff dimension δ_Γ of the limit set, following work of Akaza, Beardon, Bowen, Elstrodt, Patterson and Sullivan in the real hyperbolic case. He calculated that $\lambda_0 = \delta_\Gamma(N - \delta_\Gamma)$ where N is the Hausdorff dimension of S. On the other hand, the Laplacian on $L^2(\mathcal{H}^n/\Gamma)$ can be determined via the unitary representation of $Sp(n,1)$, say, on $L^2(Sp(n,1)/\Gamma)$. B. Kostant had determined the unitary dual of $Sp(n,1)$ and established Kazhdan's property for them with an estimate of the isolation of the trivial representation. The trivial representation of $Sp(n,1)$ is only present in $L^2(Sp(n,1)/\Gamma)$ if Γ is a lattice. Thus Corlette could apply Kostant's work to get a lower bound on λ_0 and thus an upper bound on δ_Γ, in case Γ is not a lattice. The second claim follows easily from the first, since the limit set of Γ cannot disconnect the ordinary set.

5.7 Variants of Kazhdan's property

A group G is Kazhdan if the trivial representation of G is isolated in the space of unitary representations of G. By changing the class of representations under consideration, one obtains many variants of Kazhdan's property.

Around 1980 already, M. Cowling established a stronger version of Kazhdan's property by enlarging the class of representations [34, 35]. Consider representations of a group G on a Hilbert space $\mathcal{H}$ which are not necessarily unitary but uniformly bounded in G. If G is amenable, then any uniformly bounded representation is unitarizable. For general G however, e.g. $G = SL(2,\mathbb{R})$, the two classes of representations differ.

Theorem 5.16 [Cowling, 1980] *Let G be a connected real simple Lie group with finite center. Then the trivial one-dimensional representation of G is isolated within the uniformly bounded irreducible representations precisely when the real rank of G is at least 2.*

A. Lubotzky and R. J. Zimmer investigated various weakenings of Kazhdan's property for discrete groups G by decreasing the class of representations [122]. For example, consider $G = SL(n,\mathbb{Q})$ for $n > 2$. Then G is not Kazhdan since it is not finitely generated. On the other hand, Lubotzky and Zimmer showed that the trivial representation is isolated both within the class of finite dimensional unitary representations as well as the class of unitary representations whose matrix coefficients vanish at infinity. More generally, this holds for irreducible lattices Γ in the product of a noncompact simple group with a semisimple Kazhdan group. They obtained the following geometric consequence.

Theorem 5.17 [Lubotzky-Zimmer, 1989] *Isometric ergodic actions on closed manifolds by lattices Γ as above are infinitesimally rigid.*

6 Miscellaneous Applications

6.1 Isospectral rigidity

There are now many examples of Riemannian manifolds, both in positive and negative curvature whose Laplacians have the same spectrum but which are not isometric. Negatively curved manifolds however exhibit somewhat more isospectral rigidity . To begin with, V. Guillemin and D. Kazhdan established deformation rigidity [85]. A Riemannian manifold has *simple length spectrum* if the ratio of the lengths of any two distinct closed geodesics are irrational.

Theorem 6.1 [Guillemin-Kazhdan, 1980] *Let (M, g_0) be a negatively curved closed surface with simple length spectrum. Then any deformation g_t of g_0 such the spectrum of the Laplace operator is independent of t is trivial.*

Guillemin and Kazhdan generalized this theorem later to higher dimensional manifolds with sufficiently pinched negative curvature [86]. The proof involves analyzing the representations of $O(n)$ on the tangent bundle and A. N. Livshitz' theorem on the cohomology of the geodesic flow.

Analysis of the length spectrum of a Riemannian manifold M, i.e. the set of lengths of closed geodesics is closely related with the spectrum of the Laplacian. If M is negatively curved, then each free homotopy class of loops contains precisely one closed geodesic. Assigning its length to the homotopy class defines a function from $\pi_1(M) \to \mathbb{R}$, the so-called *marked length spectrum* of M. While the length spectrum itself is not rigid, J. P. Otal and independently C. Croke showed in [153, 38]

Theorem 6.2 [Otal, Croke, 90] *The marked length spectrum determines a closed surface of negative curvature up to isometry.*

This has been generalized to nonpositively curved surfaces in [39]. Little is known in higher dimension.

Let me note here that the L^2-spectrum of an action of a semisimple group does not determine the action even measurably [188]. The construction of the counterexamples is based on Sunada's ideas on the spectral non-rigidity of the Laplace operator on a Riemannian manifold.

Theorem 6.3 [Spatzier, 1989] *Let G be a noncompact almost simple connected classical group of real rank at least 27. Then G has properly ergodic actions which are not measurably conjugate and have the same L^2-spectrum. Moreover, the actions can be chosen such that the L^2 of the spaces decomposes into a countable direct sum of irreducible representations of G.*

Let Γ be a lattice in G. It follows from Theorem 3.12 that not even the restrictions of the above purely atomic actions to Γ are measurably isomorphic.

Guillemin's and Kazhdan's work suggests the following problem.

Problem 6.4 *Let G be a semisimple group without compact and $PSL(2, \mathbb{R})$-factors. Are volume preserving actions of G and its lattices locally isospectrally rigid (or deformation rigid)?*

At least for the natural homogeneous actions, Guillemin's and Kazhdan's techniques may be helpful.

6.2 Entropy rigidity

Given a closed Riemannian manifold M, there are two fundamental measures of the complexity of its geodesic flow, namely its topological entropy h_{top} and its metric entropy h_λ with respect to the Liouville measure λ. If the sectional curvature of M is non-positive, the topological entropy can be interpreted as the exponential rate of growth of the volume of balls in the universal cover. Furthermore, for negatively curved M, there is a unique measure, called the *Bowen-Margulis measure*, whose metric entropy coincides with the topological entropy [128]. In general, the topological entropy always majorizes any metric entropy. Naturally, one asks when $h_{top} = h_\lambda$. A. Katok showed in 1982 that the metric and topological entropy of the geodesic flow of a closed surface with negative curvature coincide precisely when the surface has constant curvature [107, 108]. For higher dimensions, he conjectured that the entropies are equal if and only if the manifold is locally symmetric. He showed that this holds within the conformal class of a locally symmetric metric.

Extremal properties of the entropies are closely related. Given a closed locally symmetric space M with maximal sectional curvature -1, P. Pansu showed in 1989, using quasi-conformal methods, that any other metric on M with sectional curvature bounded above by -1 is at least as big as the topological entropy of the locally symmetric metric [157]. In 1990, U. Hamenstaedt characterized the extremal such metrics as the locally symmetric metrics amongst these [87].

Theorem 6.5 [Hamenstadt, 1990] *Let M be a closed locally symmetric space with maximal sectional curvature -1. Then the locally symmetric metrics on M are precisely the metrics which minimize the topological entropy amongst all metrics with upper bound -1 for the sectional curvature.*

Normalizing the volume rather than the maximal sectional curvature, M. Gromov conjectured that the locally symmetric metrics again are precisely the metrics which minimize the topological entropy [77]. A partial answer was found by G. Besson, G. Courtois and S. Gallot [20, 19, 20]. They endow the space of metrics on a manifold with the Sobolev topologies H^s, $s > 0$.

Theorem 6.6 [Besson-Courtois-Gallot, 1991] *Let g_0 be a metric of constant curvature -1 on a closed manifold M of dimension n. Then for all $s > n/2$, there is an H^s-neighborhood $\mathcal{U}$ of g_0 in the space of metrics with volume 1 on which the topological entropy is minimal at g_0. Furthermore, any other metric in $\mathcal{U}$ with minimal entropy has constant curvature -1.*

In fact, they can choose $\mathcal{U}$ to be a neighborhood of the conformal class of g_0, saturated by conformal classes.

L. Flaminio established a version of this theorem for C^2-deformations of g_0 transversal to the orbit of g_0 under the diffeomorphism group, using representation theory [60]. This allows him to get explicit estimates of the second derivative of the topological entropy at the constant curvature metric in terms of the L^2-norm of the variation of the metric. Interestingly, the metric entropy h_λ is neither maximized nor minimized at the constant curvature metric [60]. However, Flaminio showed that the difference $h_{top} - h_\lambda$ is convex near the constant curvature metric. He thus obtained a local entropy rigidity theorem resolving Katok's conjecture affirmatively locally [60].

Theorem 6.7 [Flaminio, 1992] *Let (M, g_0) be a closed manifold of constant negative curvature. Then along any sufficiently short path of C^∞-metrics g_t starting at g_0 and transverse to the orbit of g_0 under the diffeomorphism group, equality of topological and metric entropy implies constant curvature.*

In the proof, Flaminio calculated the derivatives of the topological and metric entropies at g_0 using the representation theory of $SO(n, 1)$. This is based on Guillemin's and Kazhdan's work on isospectral rigidity (cf. Section 6.1). To obtain precise quantitative results on the size of the derivatives, Flaminio used the full knowledge of the unitary dual of $SO(n, 1)$.

Finally let us mention a third canonical measure on a closed manifold of negative curvature, the so-called *harmonic measure*. D. Sullivan conjectured that such a space is locally symmetric provided the harmonic measure coincides with the Liouville measure. C. Yue made substantial progress in this direction [211]. Let M be a closed manifold of negative sectional curvature. Let ν_x be the harmonic measure on the ideal boundary $\tilde{M}(\infty)$ of the universal cover $\tilde{M}$ of M, i.e. the hitting probability measure for Brownian motion starting at $x \in \tilde{M}$. Let m_x be the push forward of Lebesgue measure on the unit tangent sphere S_x at x using the canonical projection from S_x to $\tilde{M}(\infty)$. Similarly, one can project the Bowen-Margulis measure of maximal entropy to measures μ_x on $\tilde{M}(\infty)$. Combining Yue's principal result with Theorem 2.17, we have

Theorem 6.8 [Yue, 1992] *The horospheres in $\tilde{M}$ have constant mean curvature provided that either $m_x = \nu_x$ for all $x \in \tilde{M}(\infty)$ or $\nu_x = \mu_x$ for all $x \in \tilde{M}(\infty)$. In either case, the geodesic flow is C^∞-conjugate to that of a locally symmetric space. Furthermore, M has constant curvature if its dimension is odd.*

6.3 Unitary representations with locally closed orbits

An action of a topological group G on a Borel space X is called *smooth* or *tame* if the quotient space X/G is countably separated. By work of J. Glimm and E. G. Effros, an action on a complete separable metrizable space X is tame if and only if all orbits are locally closed [73, 50]. The tameness of certain actions has proved very useful in rigidity theory. In the proof of Margulis' superrigidity theorem for example, the tameness of the action of a real semisimple group G on the space of probability measures on the maximal boundary H/P_H of a second group H allowed us to construct an equivariant map from the maximal boundary G/P of G to a homogeneous space of H.

R. J. Zimmer showed that unitary representations π of real algebraic groups are typically tame [215]. Define the *projective kernel*, P_π, of π by

$$P_\pi = \{g \in G \mid \pi(g) \text{ is a scalar multiple of } 1\}.$$

Theorem 6.9 [Zimmer, 1978] *If π is an irreducible unitary representation of a real algebraic group G, then π is tame provided that $\pi(P_\pi)$ is closed.*

As explained in Section 5.2, Zimmer applied this theorem to Lorentz actions.

Notice that the regular representation of a discrete group is always tame. This observation motivated the proof of the following theorem.

Theorem 6.10 [Adams-Spatzier, 1990] *Let a discrete Kazhdan group G act ergodically on a measure space S preserving a finite measure. Suppose $\beta : G \times S \to H_1 *_{H_3} H_2$ is a cocycle into an amalgamated product. Then β is cohomologous to a cocycle into H_1 or H_2.*

Amalgamated products act on trees. One can associate unitary representations of G to these trees which have almost invariant vectors and are tame. Combining this with Kazhdan's property and Zimmer's techniques from his proof of the cocycle superrigidity theorem yields the proof. The theorem easily generalizes to automorphism groups of real trees.

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