CONTINUITY IN ENRICHED CATEGORIES AND METRIC MODEL THEORY

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Dedicated to my parents and my sister, who have had unwavering faith in me.

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ABSTRACT

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We explore aspects of continuity as they manifest in two separate settings - metric model theory (continuous logic) and enriched categories - and interpret the former into the latter. One application of continuous logic is in proving that certain convergence results in analysis are in fact uniform across the choices of parameters: Avigad and Iovino [5] outline a general method to deduce from a given convergence theorem that the convergence is uniform in a "metastable" sense. While convenient, this method imposes strict requirements on the kinds of theorems allowed: in particular, any functions occurring in the theorem must be uniformly continuous. In aiming to apply to a broader class of examples the Avigad-Iovino approach, we construct a variant of continuous logic that is able to handle discontinuous functions in its domain of discourse. This logic weakens the usual continuity requirements for functions, but compensates by introducing a notion of "linear structure" that mimics e.g. the vector space structure of Banach spaces. We use this logic to apply the Avigad-Iovino method to specific convergence results from functional analysis involving discontinuous functions, and obtain uniform metastable convergence in those examples. This is the project of the first part of this thesis.

The second part of the thesis continues this study of continuity from a different angle, starting from [27] where Lawvere shows that enriching a category over $\mathbb{R}$ with the appropri-

ate monoidal structure turns that category into a metric space. He even muses on the notion of an "$\mathbb{R}$-valued logic", but does not make the connection to continuous logic (primarily because continuous logic did not yet exist). We introduce necessary structure that enables us to have a notion of "uniform continuity" and "continuous subobjects" in an enriched categorical setting, and use this to give an interpretation of continuous logic into a certain category of $\mathbb{R}$-enriched categories.

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Part I

A variant of continuous logic

Chapter 1

Introduction

Metric model theory is the study of the model-theoretic properties of metric spaces and uniformly continuous maps between them, as compared to classical model theory, which is the study of the model-theoretic properties of sets and set functions between them. There are a few (essentially equivalent) formulations of the logic interpreted by metric model theory, continuous logic being one such. We will explore how modifying certain continuity requirements in continuous logic can broaden the applicability of metric model theory to results in analysis.

Kohlenbach and others ([3], [4], [10], [16], [17], [19], [21]) have applied “proof mining” techniques to various convergence and fixed point existence results found in e.g. functional analysis to extract computable and uniform bounds from proofs that do not a priori provide such information. Here “uniform” is taken to mean “uniform in the specific functions/operators and the spaces on which they act”.

Motivated by these earlier approaches, Avigad (one of the authors of [3], [4]) and Iovino used [5] the model-theoretic machinery of continuous first order logic, in which a metric on the space replaces the equality predicate, to show at least the existence of such uniformity in many of the cases to which Kohlenbach's proof mining technique applies.

On the one hand, the Avigad-Iovino approach is more conveniently accessible to mathematicians working in fields other than logic. On the other, the continuous logic framework powering this elegant approach imposes rather stringent uniform continuity requirements on its objects of discourse. Indeed, Kohlenbach notes (for example in [18]) two advantages of his own method: one, that his proof mining is able to provide, in fact compute, the actual uniform bound, and two, the proof mining method is in a sense more robust in that it can treat cases in which the function or operator in question may have some desirable properties but is possibly discontinuous.

It is this second point that we address: we construct a variant of continuous logic (which we term *geodesic logic*) that weakens in a precise sense the continuity requirements of continuous logic but introduces a formalized notion of "linear structure" which allows us to sufficiently compensate for the resulting loss of control in the absence of continuity. Using this new framework we are able to apply Avigad-Iovino's method to a broader class of examples, in particular cases ([9], [15], [34]) in which the function in question is allowed to be discontinuous. These examples, successfully treated via the proof-theoretic approach in [22], were previously out of reach of Avigad-Iovino's model-theoretic approach.

1.1 The Avigad-Iovino method

In order to provide context for the specific applications of geodesic logic mentioned above, we first consider the following illustration of the Avigad-Iovino method based on the usual continuous logic:

Example 1.1. Let B be a reflexive Banach space and consider an operator $T : B \to B$. For $f \in B$, we have its n^{th} ergodic average $A_n f = \frac{1}{n} \sum\limits_{m<n} T^m f$.

A version of the mean ergodic theorem states that if T is power bounded (i.e. $\exists M$ such that $||T^n|| \leq M$ for all $n \in \mathbb{N}$), then given any element f of B, the sequence $\{A_n f\}$ of ergodic averages converges.

(That is, there is some $K : \mathbb{R}_{<0} \to \mathbb{N}$ such that given any $\epsilon > 0$, for all $i, j \geq K(\epsilon)$, we have that $||A_i f - A_j f|| < \epsilon$.)

1.1.1 Metastability

One might ask if there is some sense in which the above convergence is uniform across all such spaces B, operators T, and elements f of B. If we are asking for uniformity in the sense of Cauchy convergence, i.e. for a K (in the notation of Example 1.1) that is independent of the specific choice of B, T, and f, the answer is a resounding no: it is known that this convergence can be made arbitrarily slow [25]. However, we might ask for a weaker uniformity, in the following sense:

Definition 1.2. Let $\{x_n\}$ be a sequence of points in a metric space (X, d).

Given a function $F : \mathbb{N} \to \mathbb{N}$, we say that $b_F : \mathbb{R}_{>0} \to \mathbb{N}$ is a *bound on the rate of metastability* of $\{x_n\}$ with respect to F if for each $\epsilon > 0$ there exists an $n \leq b_F(\epsilon)$ such that for all $i, j \in [n, F(n)]$, we have that $d(x_i, x_j) < \epsilon$.

If such a bound b_F exists, we say that $\{x_n\}$ *converges metastably* with respect to F.

Remark 1.3. The first explicit bounds on metastability were extracted in [20], after which many other papers in proof theory on this topic were published, among them that of Avigad, Gerhardy, and Towsner in [4]; the name "metastability" is due to Tao [36]. Logically speaking, metastability is a special case of Kreisel's no-counterexample interpretation [23], [24].

It is easy to verify that a sequence $\{x_n\}$ converges in the Cauchy sense if and only if it converges metastably with respect to every $F : \mathbb{N} \to \mathbb{N}$:

Proposition 1.4. *Let $\{x_n\}$ be a sequence of points in a metric space (X, d). The following are equivalent:*

(a) There exists some $K : \mathbb{R}_+ \to \mathbb{N}$ such that for every $\epsilon > 0$ and for all $i, j \geq K(\epsilon)$, we have that $d(x_i, x_j) < \epsilon$.

(b) For each $F : \mathbb{N} \to \mathbb{N}$, $\{x_n\}$ converges metastably with respect to F.

Proof. (a) $\Rightarrow$ (b): Given any $F : \mathbb{N} \to \mathbb{N}$, define b_F by $b_F(\epsilon) = K(\epsilon)$.

(b) $\Rightarrow$ (a): Assume that $\{x_n\}$ fails to be Cauchy convergent, i.e. there is some $\epsilon > 0$ such that for every $n \in \mathbb{N}$, we can find $i_n, j_n \geq n$ such that $d(x_{i_n}, x_{j_n}) \geq \epsilon$. Let us define $F : \mathbb{N} \to \mathbb{N}$ as $F(n) = \max(i_n, j_n)$. Then $\{x_n\}$ fails to be metastably convergent for this F. $\square$

Therefore if a convergence result (e.g. the mean ergodic theorem) guarantees convergence for a class $\mathcal{C}$ of pairs $((X, d), \{x_n\})$ satisfying certain conditions, then - despite not having uniform Cauchy convergence in the sense of having a K (in the notation of Example 1.1 and Proposition 1.4 (a)) that is uniform across all members of $\mathcal{C}$ - we might ask that, once we specify some $F : \mathbb{N} \to \mathbb{N}$, whether there is a bound b_F on the rate of metastability with respect to this F which is uniform across $\mathcal{C}$.

In the case of the mean ergodic theorem, if we restrict to certain reasonable classes $\mathcal{C}$ of Banach spaces B (e.g. the class of uniformly convex Banach spaces for a fixed modulus of uniform convexity) and ergodic averages of points in a uniformly bounded subset of each B, the question above has a positive answer, as shown in [5] using continuous logic:

Theorem 1.5. ([5])

Let $\mathcal{C}$ be any class of Banach spaces with the property that the ultraproduct of any countable collection of elements of $\mathcal{C}$ is a reflexive Banach space. For every $\rho > 0$, M, and function $F : \mathbb{N} \to \mathbb{N}$, there is a bound b such that the following holds: given any Banach space B in $\mathcal{C}$, any linear operator T on B satisfying $||T^n|| \leq M$ for every n, any $f \in B$, and any $\epsilon > 0$, if $||f||/\epsilon \leq \rho$, then there is an $n \leq b$ such that $||A_i f - A_j f|| < \epsilon$ for every $i, j \in [n, F(n)]$, where $A_k = \frac{1}{k} \sum_{m<k} T^m f$.

Notice, in particular, that the operator $T : B \to B$ above is uniformly continuous. This allows for the problem to be formalized in continuous logic, and the additional conditions on B, T, and $f \in B$ then guarantee that the above particular sequence of iterations involving T converges to a fixed point. One then finds via an argument that crucially utilizes the continuous ultraproduct (see Theorem 2.7) that there is a *uniform* bound on the metastability of this convergence that is independent of the particular choice of B, T, and f.

1.1.2 Handling cases with possible discontinuity

In [9], [15], [34] one has a similar situation except that T is in general discontinuous, and so prevents the problem from being formalized in continuous logic, which requires all functions to come with moduli of uniform continuity. Specifically, consider the following:

Example 1.6. [9], [34]

(a) Let B be a Banach space, $C \subset B$ a bounded convex subset, and $T : C \to C$ a function satisfying, for some fixed $\lambda \in (0,1)$,

$$\forall x, y \in C, \quad \lambda ||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||.$$

Then [34] shows that given any $x_1 \in C$, the sequence $\{x_n\}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n$$

satisfies $d(x_n, Tx_n) \to 0$.

(b) If in addition to the above we also have that C is compact and, for some fixed $\mu \geq 1$,

$$\forall x, y \in C, \quad d(x, Ty) \leq \mu \, d(x, Tx) + d(x, y),$$

then [9] shows that the sequence $\{x_n\}$ of (a) converges to a fixed point x of T.

One might ask for a uniform bound on the rate of metastability for the convergence $d(x_n, Tx_n) \to 0$ of (a) and for the convergence $\{x_n\} \to x$ of (b). However, this problem is not formulable in continuous logic, because T is in general discontinuous; Example 2.8 gives a simple instance of $T : C \to C$ satisfying both (a) (for $\lambda = \frac{1}{2}$) and (b) (for $\mu = 3$) [34].

The idea behind geodesic logic is first to notice that the analytic arguments in the proofs of Example 1.6 all revolve around the construction and properties of the sequence $\{x_n\}$ of "iterated linear interpolations", which only relies on the underlying vector space structure. Geodesic logic abstracts this vector space structure to a general "linear structure" defined on (pseudo)metric spaces that interacts with the (pseudo)metric as expected, while dropping the continuity requirement for functions (but not for predicates and connectives). In doing so, geodesic logic is able to (1) formalize classes of examples such as the above which depend not on continuity of functions but rather an underlying linear structure on the space, while (2) preserving all of the necessary properties of the usual continuous logic that enable the Avigad-Iovino method to apply to such examples. In particular, we obtain (general versions of) the following uniformization of Example 1.6:

Theorem 1.7.

(a) *Let B be a Banach space, $C \subset B$ a convex subset with diameter bounded above by some fixed D, and $T : C \to C$ a function satisfying the condition of Example 1.6 (a). Given any $x_1 \in C$, let $\{x_n\}$ be the sequence defined by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$. Then given $F : \mathbb{N} \to \mathbb{N}$, there is a bound b_F on the rate of metastability for the sequence $d_n = d(x_n, Tx_n)$, which is uniform across all choices of B, C, T, and x_1 satisfying the above conditions.*

(b) *If in addition to the above we have that C is totally bounded with some fixed modulus of total boundedness $\beta : \mathbb{N} \to \mathbb{N}$ and T satisfies the condition of Example 1.6 (b), then given $F : \mathbb{N} \to \mathbb{N}$ there is a bound b_F on the rate of metastability for the sequence $\{x_n\}$ which is uniform across all choices of B, C, T, and x_1 satisfying the above conditions.*

We describe geodesic logic in Chapter 3 after outlining the features of the usual continuous logic and the Avigad-Iovino method in Section 2. We then describe in detail the analytic aspects of the examples of [9], [15], [34] in Chapter 4. Finally in Chapter 5 we show that geodesic logic is indeed able to handle the relevant features of such examples, thus enabling the Avigad-Iovino approach to yield (Theorem 5.6 and Theorem 5.9) a uniform bound on the rate of metastability for the sequences in question.

We thus illustrate the applicability of geodesic logic with specific examples involving iterated linear interpolation, and given the prevalence of linear interpolation arguments in e.g. functional analysis, we expect this variant of continuous logic to meaningfully broaden the scope of the applicability of metric model theory to such disciplines.

Chapter 2

Preliminaries

2.1 Syntax and interpretation

We first describe the features of continuous first order logic relevant to our current interests: more details can be found in e.g. [6].

Definition 2.1. ([6])

Let (X, d) be a complete, bounded metric space. We have the following definitions:

(a) (i) An *(n-ary) function* f on X is a uniformly continuous function $f : X^n \to X$.

(ii) An *(n-ary) predicate* R *(with range* a*)* on X is a uniformly continuous function $R : X^n \to [0, a]$.

(iii) A *constant* c on X is an element $c \in X$.

(b) Given some (possibly empty) distinguished family $\{f_i, R_j, c_k \mid i \in I, j \in J, k \in K\}$ of functions, predicates, and constants on X, we call this data

$$\mathcal{X} = (X, d, f_i, R_j, c_k \mid i \in I, j \in J, k \in K)$$

a *metric structure.*

In the above, we always consider X^n as equipped with the maximum metric, i.e. $d_{X^n}(x, y) = \max_{1 \leq m \leq n} d(x_m, y_m)$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Given such a metric structure $\mathcal{X}$, we can talk about the *signature* corresponding to that structure, which is the collection of names of the various objects that comprise the metric structure:

Definition 2.2. ([6])

Let $\mathcal{X} = (X, d, f_i, R_j, c_k \mid i \in I, j \in J, k \in K)$. A *signature* S for $\mathcal{X}$ consists of:

(a) A symbol d corresponding to the metric d of X, and a nonnegative real number D_X specifying an upper bound for the diameter of X.

(b) For each function $f_i : X^n \to X$, a *function symbol* f_i and a modulus of uniform continuity $\delta_{f_i} : \mathbb{R} \to \mathbb{R}$ for f_i (i.e. a function δ_{f_i} such that $d_{X^n}(x, y) < \delta_{f_i}(\epsilon)$ implies $d(f_i(x), f_i(y)) < \epsilon$).

(c) For each predicate $R_j : X^n \to [0, a_j]$, a *predicate symbol* R_j (and a symbol for the range a_j of R_j) and a modulus of uniform continuity $\delta_{R_j} : \mathbb{R} \to \mathbb{R}$ for R_j.

(d) For each constant $c_k \in X$, a *constant symbol* c_k.

Conversely, if S is some signature and $\mathcal{X}$ is some metric structure for whom the symbols of S satisfy the above conditions, then $\mathcal{X}$ is called an S-*structure.*

Following the authors of [6], we will assume for simplicity's sake that $D_X = 1$ and $a_j = 1$ throughout. Also, we reserve the right to abuse notation by reusing the index set I in other contexts possibly unrelated to the above definitions.

Given a signature S - which specifies the vocabulary of the language in which we can speak - we can talk of (first-order) formulae and sentences in the language. First, we say that the *logical symbols* of S include d (which plays the role of equality in classical first-order logic, where $d(x, y) = 0$ is analogous to the classical statement $x = y$); an infinite set $V_S = \{x_i \mid i \in I\}$ of variables, for I some index set (a priori unrelated to the set indexing the function symbols of S); a symbol u for each continuous function $u : [0, 1]^n \to [0, 1]$ (which plays the role of n-ary connectives); and the symbols sup and inf which are analogous to the classical quantifies $\forall$ and $\exists$, respectively.

We then say that the *nonlogical symbols* of S are the function, predicate, and constant symbols of S. The *cardinality* $|S|$ of S is the smallest infinite number $\geq$ the cardinality of the set of nonlogical symbols of S.

Definition 2.3. ([6])

Let S be a signature.

(a) A *term* for S is given by the following inductive description:

 (i) Each variable and each constant is a term.

 (ii) $f(t_1, \dots, t_n)$ is a term when f is some (n-ary) function symbol and each t_i is itself a term.

(b) An *atomic formula* for S is given by an expression of the form $P(t_1, \dots, t_n)$ where P is some (n-ary) predicate symbol and each t_i is a term. (The symbol d for the metric is treated as a binary predicate symbol.)

(c) A *formula* for S is given by the following inductive description:

(i) Each atomic formula is a formula.

(ii) $u(\phi_1, \ldots, \phi_n)$ is a formula when u is some n-ary connective, i.e. a continuous function $[0,1]^n \to [0,1]$, and each ϕ_i is a formula.

(iii) $\sup_x \phi$ and $\inf_x \phi$ are each formulae when x is a variable and ϕ is a formula.

Many notions from classical first order logic carry over unmodified; *subformulae* of a formula and *substitution* of a term for a variable are a few examples. We then say that if a variable x occurs in a formula ϕ and x is not contained in any subformula of the form $\sup_x \phi'$ or $\inf_x \phi'$ (i.e. x is not quantified over), then x is a *free variable* in ϕ. A formula ϕ that has no free variables is called a *sentence*.

Often we will write a term t as $t(x_1, \ldots, x_n)$ to make it clear which (distinct) variables occur in t. Similarly we write a formula ϕ as $\phi(x_1, \ldots, x_n)$ to make it clear which are the (distinct) free variables occurring in ϕ.

If ϕ is a formula with no free variables, then ϕ is called a *sentence* (also called an S-sentence, when we are working within a signature S).

Given a signature S with its attendant logical and nonlogical symbols, and a correspondence between S and a metric structure $\mathcal{X}$, it is clear what the interpretation of each term and formula should be, since they are built up inductively out of functions, predicates, and constants, the interpretation of which is a priori given via the aforementioned correspondence. For complete details, see [6].

It is straightforward to verify that, from the moduli of uniform continuity of all the functions and predicates that occur in a given formula, we can find a modulus of uniform continuity for that formula.

Given two S-formulae $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$, we define their *logical distance*

$|\phi - \psi|$ as

$$|\phi - \psi| = \sup_{\mathcal{X}; x_1, \ldots, x_n \in \mathcal{X}} |\phi(x_1, \ldots, x_n) - \psi(x_1, \ldots, x_n)|$$

and ϕ, ψ are said to be *logically equivalent* when $|\phi - \psi| = 0$.

It is possible to restrict our (a priori uncountable) set of logical connectives to a more manageable, countable set of connectives with a very compact description using the above notion of logical distance and density with respect to said distance, and then to talk about "definable" predicates (and functions, subsets, etc.) - but we will not outline this direction in this paper, and instead refer the interested reader to [6] for details.

In continuous logic, we call formulae (resp. sentences) of the form $\phi = 0$ *conditions* (resp. closed conditions). These play the same role that formulae and sentence play in the usual first-order logic. If ϕ and ψ are formulae then we can regard formulae of the form $\phi = \psi$ as shorthand for the condition $|\phi - \psi| = 0$. We can thus regard formulae of the form $\phi = r$ for $r \in [0, 1]$ as a special case of this, by considering r as a 0-ary connective. In continuous logic we are usually content with models satisfying "arbitrarily close" approximations to a given condition $\phi = r$, so it suffices to restrict the set of 0-ary connectives r to $\mathbb{Q} \cap [0, 1]$.

Similarly, we can regard $\phi \leq \psi$ as the condition $\phi \dot{-} \psi = 0$, where $t_1 \dot{-} t_2 = \max(t_1 - t_2, 0)$. If Σ is a set of conditions, then we denote by Σ^+ the set of conditions $\phi \leq \frac{1}{n}$ for each $n \in \mathbb{N}$ and each formula ϕ such that $\phi = 0$ is in Σ. If Σ is a set of closed conditions, then we say that $\mathcal{X}$ is a *model* of Σ when $\mathcal{X}$ satisfies every condition in Σ, where the notion of "satisfaction" of a condition by a structure $\mathcal{X}$ is the obvious analogue of "satisfaction" as defined in usual first-order logic. Clearly $\mathcal{X}$ is a model of Σ if and only if it is a model of Σ^+.

2.2 The continuous ultraproduct

We now describe ultraproducts (in the sense of [6]), as they occupy a central role in both [5] and this paper. For completeness' sake, we start by defining ultrafilters:

Definition 2.4. Let I be a set.

(a) A (proper) *filter* on I is a set $\mathcal{F} \subset \mathcal{P}(I)$ (where $\mathcal{P}$ gives the powerset of its argument, and so $\mathcal{F} \in \mathcal{P}(\mathcal{P}(I))$) satisfying the following:

  (i) $\emptyset \notin \mathcal{F}$.

  (ii) $\mathcal{F}$ is upward closed, i.e. if $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

  (iii) $\mathcal{F}$ is closed under finite intersection, i.e. if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

(b) A (proper) filter $\mathcal{F}$ on I is an *ultrafilter* on I if for every $A \in \mathcal{P}(I)$, either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$.

The condition (b) of the above definition for a filter $\mathcal{F}$ to be an ultrafilter given above is equivalent to $\mathcal{F}$ being a maximal filter, where $\mathcal{P}(\mathcal{P}(I))$ is partially ordered with respect to inclusion. An ultrafilter $\mathcal{F}$ is called *principal* if it is the ultrafilter generated by a singleton set, i.e. $\mathcal{F} = \{A \in \mathcal{P}(I) \mid i_0 \in A\}$ for some $i_0 \in I$ (and of course, $\mathcal{F}$ is called *nonprincipal* if it is not principal). In all of our constructions involving ultrafilters, we will assume that our ultrafilter is nonprincipal.

Sometimes it is more convenient to talk of a *filter base*, where we say that $\mathcal{F}' \subset \mathcal{P}(I)$ is a base for a filter $\mathcal{F}$ (or that $\mathcal{F}'$ *generates* $\mathcal{F}$) if $\mathcal{F}'$ satisfies (a)(i) of the above definition, and it is downward closed, i.e. for $A, B \in \mathcal{F}'$, there is some $C \in \mathcal{F}'$ such that $C \subset A \cap B$. $\mathcal{F}$ is then the minimal filter containing $\mathcal{F}'$, i.e. $\mathcal{F} = \{A \in \mathcal{P}(I) \mid A' \subset A, A' \in \mathcal{F}'\}$. A popular example of a nonprincipal ultrafilter is any ultrafilter containing the cofinite filter on $\mathbb{N}$ (that is, the filter generated by the base $\{A \in \mathcal{P}(\mathbb{N}) \mid \mathbb{N} \setminus A \text{ is finite}\}$).

Before we actually define the ultraproduct construction, we should note a few facts which we will require. Let X be a topological space, and $\{x_i\}$ some family of points on X, indexed by a set I. Let $\mathcal{F}$ be an ultrafilter on I. We say that $x = \lim_{i,\mathcal{F}} x_i$ or that x is a $\mathcal{F}$-limit of the family $\{x_i\}$ when for every neighborhood U of x, we have $\{i \mid x_i \in U\} \in \mathcal{F}$. If X is Hausdorff, this limit must be unique.

Definition 2.5. ([6])

Let S be some signature, and $\mathcal{X}_i$ a family of S-structures, indexed by some set I. Let $\mathcal{F}$ be an ultrafilter on I.

Let $\tilde{X} = \prod_i X_i$ be the cartesian product of the underlying spaces of $\mathcal{X}_i$. There is an induced function $d : \tilde{X} \times \tilde{X} \to [0,1]$ given by $d((x_i),(y_i)) = \lim_{i,\mathcal{F}} d_i(x_i, y_i)$.

Let $\sim_{\mathcal{F}}$ be the equivalence relation on $\tilde{X}$ given by $x \sim y \Leftrightarrow d(x,y) = 0$, and let $X = \tilde{X}/\sim_{\mathcal{F}}$.

We call X the *$\mathcal{F}$-ultraproduct* of the spaces X_i. If all the X_i are the same, then we also call X their *$\mathcal{F}$-ultrapower.*

For each function, predicate, and constant symbol in S in the above definition, we have a family $\{f_i\}$, $\{R_i\}$, $\{c_i\}$ of functions, predicates, and constants interpreting those symbols in each $\mathcal{X}_i$. For each such family of objects, the above construction induces a corresponding ultraproduct object. That is, given a family $\{f_i : X_i \to X_i\}$ of functions, we have a function $f : X \to X$ defined as $f(x) = [(f_i(x_i))]_{\mathcal{F}}$ where (x_i) is a representative of the equivalence class of x in X and $[(f_i(x_i))]_{\mathcal{F}}$ is the equivalence class of $(f_i(x_i)) \in \prod_i X_i$. (That f is well-defined follows from the fact that all the f_i share the same modulus of uniform continuity and from the way $\sim$ is defined.) Note that this f shares the same modulus of uniform continuity with each of the f_i.

Similarly we have that the $\{R_i\}$ define a predicate R in the ultraproduct, and that the c_i define a constant c in the ultraproduct. Thus given a family of structures $\{\mathcal{X}_i\}$, we have not only an ultraproduct of their underlying spaces, but an ultraproduct of S-structures, which is itself an S-structure.

Ultraproducts feature prominently in [5] as well as this paper, in large part due to the following variant of Łos's theorem, the moral content of which is that "a statement is true of the ultraproduct if and only if it is mostly true of its factors."

Theorem 2.6. ([6])

Let S be a signature, and $\{\mathcal{X}_i\}$ an I-index family of S-structures. Let $\mathcal{F}$ be an ultrafilter on I, and $\mathcal{X}$ the $\mathcal{F}$-ultraproduct of the $\{\mathcal{X}_i\}$ having X as its underlying space.

Let $\phi(x)$ be an S-formula, with $\{a_i\}$ a family of elements of X_i. Let a be the corresponding element in X. Then:

$$\phi(a) = \lim_{i,\mathcal{F}} \phi(a_i)$$

The proof of the above theorem, which is actually more general (it is true of formulae ϕ depending on any number n of free variables), is through induction on the complexity of formulae.

Our (and [5]'s) interest in Theorem 2.6 lies in leveraging it to obtain the following theorem (due to [5] but rephrased slightly here to better reflect the underlying logical machinery), which is the main ingredient of the proof of Theorem 1.5:

Theorem 2.7. ([5])

Let S be a signature, and let $\{t_n\}$ be a sequence of S-terms.

Let $\mathcal{C}$ be a collection of S-structures $\mathcal{X}$, and for each $\mathcal{X}$ let $\{x_n\}$ denote the interpretation in $\mathcal{X}$ of the sequence $\{t_n\}$.

Finally, let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. Then the following are equivalent:

(a) For every $\epsilon > 0$ and every $F : \mathbb{N} \to \mathbb{N}$, there is some $b \geq 1$ such that the following holds: for every $\mathcal{X}$ in $\mathcal{C}$, there is an $n \leq b$ such that $d(x_i, x_j) < \epsilon$ for every $i, j \in [n, F(n)]$.

(b) For any sequence $\{\mathcal{X}_k\}$ of elements of $\mathcal{C}$, let $\mathcal{X}$ be their $\mathcal{F}$-ultraproduct. Then for every $\epsilon > 0$ and every $F : \mathbb{N} \to \mathbb{N}$, there is an n such that $d(x_i, x_j) < \epsilon$ for every $i, j \in [n, F(n)]$.

Proof. (a) $\Rightarrow$ (b): For any fixed $\frac{1}{2}\epsilon > 0$ and any fixed $F : \mathbb{N} \to \mathbb{N}$, there is some $b \geq 1$ such that every member $\mathcal{X}$ of $\mathcal{C}$ satisfies the condition

$\min_{n \leq b} \left(\max_{i,j \in [n,F(n)]} d(x_i, x_j) \right) \leq \frac{1}{2}\epsilon$, or more formally,

$\min_{n \leq b} \left(\max_{i,j \in [n,F(n)]} (d(x_i, x_j) \dot{-} \frac{1}{2}\epsilon) \right) = 0.$

Since every member of $\mathcal{C}$ is a model of the above condition, any ultraproduct of members of $\mathcal{C}$ must again be a model of this condition.

(b) $\Rightarrow$ (a): If for some $\epsilon > 0$ and some $F : \mathbb{N} \to \mathbb{N}$ there is no bound b such as in (a), then for each $k \in \mathbb{N}$, there is some $\mathcal{X}_k \in \mathcal{C}$ that is a counterexample to k being such a bound. That is, for each $\mathcal{X}_k$, there is an $n \leq k$ such that $d_k(x_i^k, x_j^k) \geq \epsilon$ for some $i, j \in [n, F(n)]$. Let $\mathcal{X}$ be the $\mathcal{F}$-ultraproduct of these structures $\mathcal{X}_k$.

Given any n, since there are cofinitely many $k \geq n$, there are cofinitely many k such that there exist $i, j \in [n, F(n)]$ with $d_k(x_i^k, x_j^k) \geq \epsilon$. It follows that there is some specific pair $i, j \in [n, F(n)]$ such that $d_k(x_i^k, x_j^k) \geq \epsilon$ for $\mathcal{F}$-many k, so that $d(x_i, x_j) = \lim_{k,\mathcal{F}} d_k(x_i^k, x_j^k) \geq \epsilon$

for that choice of i, j. Since n was arbitrary, we see that (b) fails. $\square$

2.3 The role of continuity

The starring role of the continuous ultraproduct in this crucial theorem illustrates why uniform continuity is necessary in applying the Avigad-Iovino approach to obtaining uniformity. Let us consider a toy example due to [34] which shows what can happen in the absence of uniform continuity:

Example 2.8. ([34])

Let $T : [0, 3] \to [0, 3]$ be defined by $Tx = \begin{cases} 0 & \text{for } x \neq 3 \\ 1 & \text{for } x = 3 \end{cases}$.

Let $\mathcal{F}$ be an ultrafilter containing the cofinite filter on $\mathbb{N}$, and let $([0, 3])_{\mathcal{F}}$ denote the ultrapower of $[0, 3]$ with respect to this ultrafilter.

The sequence $\{a_n = 3\}$ represents the same point in the ultrapower as the sequence $\{b_n = 3 - \frac{1}{n}\}$, while the sequences $\{Ta_n = 1\}$ and $\{Tb_n = 0\}$ represent different points. That is, the ultrapower of the function T fails to be well-defined. (This kind of phenomenon is precisely what having a uniform modulus of continuity would prevent.)

Although the function given in Example 2.8 is discontinuous, it is an instance of a function that is well behaved in other ways:

Definition 2.9. ([34])

Let X be a Banach space and C a nonempty subset. A function $T : C \to X$ is said to satisfy *condition (C)* when for all $x, y \in C$,

$\frac{1}{2}||x - Tx|| \leq ||x - y||$ implies $||Tx - Ty|| \leq ||x - y||$.

Any nonexpansive mapping satisfies condition (C), but condition (C) is clearly weaker. For instance, it is easily verified that the function in Example 2.8 satisfies condition (C). [9] and [34] show how this condition can be leveraged, in the presence of certain other topological conditions, to yield the existence of a fixed point to which a certain kind of iteration sequence converges - we will describe this in detail in Chapter 4. The point is that a function might be discontinuous yet satisfy conditions that guarantee convergence to a fixed point, and so we might ask if the Avigad-Iovino approach to showing that such convergence is uniform (in the sense of Theorem 1.5) could be adapted to settings in which the objects in question are allowed to be discontinuous yet are nevertheless "nice" in other ways, given the relative convenience of said approach.

We will show that this is indeed possible, by making slight modifications to the framework of continuous logic. One particular modification is to weaken the equivalence relation we quotient by when taking the ultraproduct. The usual equivalence relation forces the resulting ultraproduct to be a strict metric space (which leads to problems of the type we have seen above), while our modification produces an ultraproduct which is only a pseudometric space in general. In order that these modifications preserve the full strength of the usual continuous logic e.g. in applications to fixed point theory of Banach spaces, we must show that the relevant convergence proofs (which were given in the context of complete *normed* vector spaces, which are in particular *metric* spaces with a kind of hyperbolic linear structure) actually apply to the pseudometric case as well, in many cases with very little or even no modification.

Chapter 3

The general setup

We begin with a necessary definition:

Definition 3.1. Let X be a set. A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is a *pseudometric* for X when it satisfies the following conditions:

(a) $\forall x \in X, d(x, x) = 0.$

(b) $\forall x, y \in X, d(x, y) = d(y, x).$

(c) $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z).$

By a *pseudometric space* we refer to a pair (X, d) where X is a set and d is a pseudometric for X.

Remark 3.2. We will frequently have occasion to talk about *bounded* pseudometric spaces, i.e. spaces (X, d) where the pseudometric d takes values in some bounded interval $[0, D]$ for some positive real number D. We call D a *bound* for the space X, and by abuse of notation we may consider $d : X \times X \to \mathbb{R}_{\geq 0}$ as instead a function $d : X \times X \to [0, D]$.

Note that every pseudometric space is naturally a topological space (the set of ϵ-balls $\{y \mid d(x,y) = \epsilon\}$ for each $x \in X$ and each $\epsilon > 0$ is a basis for the topology on X associated with the pseudometric), and that pseudometric spaces are general enough to include normed vector spaces (in particular, Banach spaces) as a special case.

3.1 Geodesic logic

As continuous logic is built upon the theory and properties of metric spaces as its foundation, our modified continuous logic will have pseudometric spaces as its foundation. We will often find that the pseudometric spaces we are interested in have additional structure (e.g. vector space structure) which features meaningfully in our investigations of them. We note one particular type of such structures, which is a generalization of the vector space structure of a normed vector space.

Definition 3.3. A pseudometric space (X, d) is said to be equipped with a *linear structure* L when there is a specified function $L : X \times X \times [0,1] \to X$ satisfying the following:

(a) For each pair $x, y \in X$ of points, the map $L(x, y, \frac{1}{d(x,y)}(\,\cdot\,)) : [0, d(x,y)] \to X$ is an isometric embedding (i.e. a geodesic between x and y).

(b) $d(L(x,y,t), L(y,x,1-t)) = 0$.

If (X, d) is a pseudometric space with linear structure L we will sometimes refer to it as (X, d, L); and when the context is clear we might say that "X is a space with linear structure."

Given these notions, let us now describe the basics of our modified continous logic:

Definition 3.4. Let (X, d) be a complete, bounded pseudometric space. We have the following:

(a) (i) An *(n-ary) function* T on X is a (possibly discontinuous) function $T : X^n \to X$.

(ii) An *(n-ary) continuous function* f on X is a uniformly continuous function $f : X^n \to X$.

(iii) An *(n-ary) predicate* R *(with range* a*)* on X is a uniformly continuous function $R : X^n \to [0, a]$.

(iv) A *constant* c on X is an element $c \in X$.

(b) Given some (possibly empty) distinguished family

$$\{T_i, f_{i'}, R_k, c_l \mid i \in I, i' \in I', j \in J, k \in K\}$$

of functions, continuous functions, predicates, and constants on X, we call this data $\mathcal{X} = (X, d, T_i, f_{i'}, R_j, c_k \mid i \in I, i' \in I', j \in J, k \in K)$ a *pseudometric structure.*

Again, as in Definition 2.1, we consider X^n in Definition 3.4 above as equipped with the maximum pseudometric given by $d_{X^n}(x, y) = \max_{1 \leq m \leq n} d(x_m, y_m)$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Given a pseudometric structure $\mathcal{X}$, we can talk of the corresponding signature:

Definition 3.5. Let $\mathcal{X} = (X, d, T_i, f_{i'}, R_j, c_k \mid i \in I, i' \in I', j \in J, k \in K)$. A *signature* S for $\mathcal{X}$ consists of:

(a) A symbol d corresponding to the metric d of X, and a nonnegative real number D_X specifying an upper bound for the diameter of X.

(b) For each function $T_i : X^n \to X$, a *function symbol* T_i.

(c) For each continuous function $f_{i'} : X^n \to X$, a *continuous function symbol* f_i and a modulus of uniform continuity $\delta_{f_i} : \mathbb{R} \to \mathbb{R}$ for f_i (i.e. a function δ_{f_i} such that $d_{X^n}(x, y) < \delta_{f_i}(\epsilon)$ implies $d(f_i(x), f_i(y)) < \epsilon$).

(d) For each predicate $R_j : X^n \to [0, a_j]$, a *predicate symbol* R_j (and a symbol for the range a_j of R_j) and a modulus of uniform continuity $\delta_{R_j} : \mathbb{R} \to \mathbb{R}$ for R_j.

(e) For each constant $c_k \in X$, a *constant symbol* c_k.

Conversely, if S is some signature and $\mathcal{X}$ is some pseudometric structure for whom the symbols of S satisfy the above conditions, then $\mathcal{X}$ is called an *S-structure*.

With the exception of the ultraproduct, everything else not specifically mentioned above (e.g. terms, formulae, connectives, etc.) remains unchanged from the usual continuous logic. We defer the description of the ultraproduct until Section 3.2. Let us call this variant of continuous logic "optionally continuous logic" (OCL for short). OCL is not very interesting, since although it is technically a generalization of continuous logic, it is essentially a regression back towards classical (non-continuous) first-order logic.

However, we can introduce additional structure (namely, linear structure) to OCL to compensate for the control that we lose by allowing for discontinuous functions. Let us call the resulting variant *geodesic logic*:

Definition 3.6. Let $\mathcal{X}' = (X, d, T_i, f_{i'}, R_j, c_k \mid i \in I, i' \in I', j \in J, k \in K)$ be a pseudometric structure, and let L be a linear structure on (X, d).

For each $t \in [0, 1]$, let L_t be the function $X \times X \to X$ defined by $L_t(x, y) = L(x, y, t)$. Call each L_t the *t-value of the linear structure L.*

We call $\mathcal{X} = (X, d, L_t, T_i, f_{i'}, R_j, c_k \mid t \in [0,1], i \in I, i' \in I', j \in J, k \in K)$ a *geodesic structure.*

Furthermore, a *(geodesic) signature* S for $\mathcal{X}$ consists of:

(a) A symbol d corresponding to the metric d of X, and a nonnegative real number D_X specifying an upper bound for the diameter of X.

(b) For each $t \in [0,1]$, a *t-linear structure symbol* L_t corresponding to the t-value L_t of the linear structure L of X.

(c) For each function $T_i : X^n \to X$, a *function symbol* T_i.

(d) For each continuous function $f_{i'} : X^n \to X$, a *continuous function symbol* f_i and a modulus of uniform continuity $\delta_{f_i} : \mathbb{R} \to \mathbb{R}$ for f_i (i.e. a function δ_{f_i} such that $d_{X^n}(x,y) < \delta_{f_i}(\epsilon)$ implies $d(f_i(x), f_i(y)) < \epsilon$).

(e) For each predicate $R_j : X^n \to [0, a_j]$, a *predicate symbol* R_j (and a symbol for the range a_j of R_j) and a modulus of uniform continuity $\delta_{R_j} : \mathbb{R} \to \mathbb{R}$ for R_j.

(f) For each constant $c_k \in X$, a *constant symbol* c_k.

Conversely, if S is some geodesic signature and $\mathcal{X}$ is some geodesic structure for whom the symbols of S satisfy the above conditions, then $\mathcal{X}$ is called an *S-structure.*

In what follows, we will assume as before, again following the authors of [6], that the bound for our spaces and the codomains of predicates are just the interval $[0,1]$.

As was the case for OCL, with the exception of the ultraproduct (which again, we will describe in Section 3.2), most concepts not specifically mentioned above carry over unchanged from continuous logic. However, for the sake of completeness, we describe terms and formulae in geodesic logic:

Definition 3.7. Let S be a geodesic signature.

(a) A *term* for S is given by the following inductive description:

  (i) Each variable and each constant is a term.

  (ii) $T(t_1, \ldots, t_n)$ is a term when T is some (n-ary) function symbol and each t_i is itself a term.

  (iii) $f(t_1, \ldots, t_n)$ is a term when f is some (n-ary) continuous function symbol and each t_i is itself a term.

  (iv) For each $t \in [0, 1]$, $L_t(t_1, t_2)$ is a term when L_t is the t-linear structure symbol and t_1, t_2 are terms. (That is, L_t is treated as a binary function symbol.)

(b) An *atomic formula* for S is given by an expression of the form $P(t_1, \ldots, t_n)$ where P is some (n-ary) predicate symbol and each t_i is a term. (The symbol d for the pseudometric is treated as a binary predicate symbol.)

(c) A *formula* for S is given by the following inductive description:

  (i) Each atomic formula is a formula.

  (ii) $u(\phi_1, \ldots, \phi_n)$ is a formula when u is some n-ary connective, i.e. a continuous function $[0, 1]^n \to [0, 1]$, and each ϕ_i is a formula.

  (iii) $\sup_x \phi$ and $\inf_x \phi$ are each formulae when x is a variable and ϕ is a formula.

Remark 3.8. The reason for treating the linear structure $L : X \times X \times [0,1] \to X$ as consisting of separate functions $L_t : X \times X \to X$ is that, due to the specific technicalities of the geodesic ultraproduct (which will be addressed in Section 3.2), it is problematic to regard L as simply another function symbol L of OCL. If we were to incorporate L itself as a function symbol, the relationship between *the symbol* L and its interpretation as a linear structure L on X would have to be distinct from that between some function symbol f and its interpretation as a function $f : X \times X \times [0,1] \to X$, in precisely the manner that has been built in to Definition 3.6 by considering the linear structure L as a family of functions L_t, each of which then receives the same treatment (e.g. under ultraproducts) as the other function symbols do under OCL.

However, when we speak informally of linear structures for geodesic structures and there is no possibility for confusion, we will usually speak of L rather than the family L_t for convenience.

We note that geodesic structures (with notation as Definition 3.6) can be characterized in OCL by the following axioms:

(a) For each pair $t, t' \in [0,1]$,

  (i) $\sup_x \sup_y \left(d(L_t(x,y), L_{t'}(x,y)) \dot{-} |t-t'|\, d(x,y)\right) = 0$, and

  (ii) $\sup_x \sup_y \left(|t-t'|\, d(x,y) \dot{-} d(L_t(x,y), L_{t'}(x,y))\right) = 0$

(b) For each $t \in [0,1]$, $\sup_x \sup_y \left(d(L_t(x,y), L_{1-t}(y,x))\right) = 0$.

It is clear from Definition 3.7 that for S a geodesic signature, if ϕ is either an S-term or S-formula containing only continuous function symbols and predicate symbols (and connectives, which we require to be the same as in continuous logic), then ϕ will also have a modulus of uniform continuity.

A given t-value L_t of a linear structure L need not be continuous in its arguments. However, there is an interesting particular class of spaces with linear structure satisfying a different niceness condition, as an example of an axiomatizable class in geodesic logic:

Definition 3.9. ([11], [35]) A pseudometric space with linear structure (X, d, L) is of *hyperbolic type* when for each quadruple $p, x, y, m \in X$ of points where $m = L(x, y, t)$ for some $t \in [0, 1]$, we have that $d(p, m) \leq (1-t)\, d(p, x) + t\, d(p, y)$.

That spaces of hyperbolic type are axiomatizable in geodesic logic follows from the easy observation below:

Let S be a geodesic signature, with linear structure L. Then an S-structure $\mathcal{X} = (X, d, L, \dots)$ is of hyperbolic type if and only if $\mathcal{X}$ satisfies, for each $t \in [0, 1]$, the S-condition

$$\sup_p \sup_x \sup_y ((d(p, L_t(x, y)) \dot{-} (1-t)\, d(p, x)) \dot{-} t\, d(p, y)) = 0. \tag{3.1}$$

This condition, as its name suggests, is a notion of hyperbolicity for pseudometric spaces which is general enough to include e.g. CAT(0) spaces as a special case.

One should note that in general, a space may possess many different linear structures, and specifying a geodesic structure on the space simply picks out a favored linear structure. Being of hyperbolic type ensures that this linear structure is nice in the sense of Definition 3.9, but is a priori a property only of the specified linear structure. In particular, being of hyperbolic type does not preclude the existence of other linear structures; thus it is a weaker condition than many other "versions" of hyperbolicity which either imply or explicitly require unique geodesicity (i.e. uniqueness and existence of isometric embeddings of line segments between points).

Indeed, every Banach space (with linear structure given by its vector space structure) is a space of hyperbolic type, while there are many Banach spaces which are not uniquely geodesic and therefore possess multiple linear structures:

Example 3.10. Consider $\mathbb{R}^2$ with the supremum (maximum) norm. Between the points $(0,0)$ and $(2,0)$, there is the obvious geodesic $t \mapsto (t,0)$. However, we can find another geodesic between them given by the piecewise map $t \mapsto (t,t)$ for $t \leq 1$ and $t \mapsto (t,2-t)$ for $t > 1$.

Thus this Banach space has at least two possible linear structures: one given by the standard vector space structure (call it L), and another L' defined by

$$L'(x,y,t) = \begin{cases} L(x,y,t) & \text{for } x,y \notin \{(0,0),(2,0)\} \\ \begin{cases} (t,t) & \text{for } t \leq 1 \\ (t,2-t) & \text{for } t > 1 \end{cases} & \text{for } x = (0,0), y = (2,0) \end{cases}$$

(and $L'(x,y,t) = L'(y,x,1-t)$).

Spaces of hyperbolic type therefore comprise a quite general class of spaces; we give an example of a space with linear structure that fails to be of hyperbolic type:

Example 3.11. Consider S^2 with its standard metric. Between any pair of non-antipodal points there is a unique geodesic, and between any pair of antipodal points we can simply pick a geodesic (subject to the symmetry condition of Definition 3.3 (b)), giving us a linear structure on S^2 considered as a pseudometric space.

Fix a point $p \in S^2$ as the "north pole", along with a pair of (necessarily non-antipodal) distinct points $x, y \in S^2$ in the open southern hemisphere lying on the same latitude. Let m be the point halfway on the geodesic between x and y. Then $d(p, m) > \frac{1}{2}d(p, x) + \frac{1}{2}d(p, y)$ since $d(p, x) = d(p, y)$ and m lies on the great circle between x and y.

The phenomenon described in the example above must happen for *any* linear structure on S^2 with its standard metric (due to unique geodesicity between non-antipodal points); thus S^2 with its standard metric cannot possess any linear structure that makes it a space of hyperbolic type.

The useful property of a space being of hyperbolic type is thus easily translated into the framework of geodesic logic. There are, however, important properties involving the linear structure of a space which are not as readily translated:

Definition 3.12. We say that a subset $C \subset X$ is *convex* (with respect to the linear structure L) when for all $x, y \in C$ and for all $t \in [0, 1]$, $L(x, y, t) \in C$.

The property of a subset being convex depends on the specific linear structure: let us again consider the Banach space $\mathbb{R}^2$ of Example 3.10. Letting $C = [0, 2] \times \{0\} \subset \mathbb{R}^2$, clearly C is convex with respect to L but not with respect to L'.

This is not the only issue with the notion of convexity. Trying to formalize convexity within the framework of continuous logic (of either the usual or our modified kind) leads immediately to at least the following two questions: (1) how do we formalize the notion of subset, and (2) how do we deal with implication, which is essentially a discontinuous connective, in a logic that only allows uniformly continuous connectives?

The first question has the following answer:

Let $C \subset X$ be a closed subset of a pseudometric space X. We can consider a predicate $\hat{C} : X \to [0,1]$ defined as $\hat{C}(x) = d(x, C) = \inf_{y \in C} d(x, y)$, so that $C = \{x \mid \hat{C}(x) = 0\}$. It turns out that these kinds of predicates have a nice characterization, the proof of which is irrelevant to our purposes so we refer to interested reader to [6]:

Proposition 3.13. ([6])

If a predicate $P : X \to [0,1]$ is of the form $P(x) = d(x, C)$ for some subset $C \subset X$, then it satisfies the following statements which we collectively refer to as subsets-as-predicates axioms *(or* s.a.p. axioms *for short):*

(a) $\sup_x \inf_y \max(P(y), |P(x) - d(x, y)|) = 0$

(b) $\sup_x |P(x) - \inf_y \min(P(y) + d(x, y), 1)| = 0$

Conversely, if a given predicate $P : X \to [0,1]$ satisfies the s.a.p. axioms, then it is of the form $P(x) = d(x, C)$ where $C = \{x \mid P(x) = 0\}$. Thus there is a one-to-one correspondence between closed subsets of X and predicates on X satisfying the s.a.p. axioms.

This correspondence between closed subsets and predicates is what will allow us to (by abuse of notation) speak of them interchangeably without confusion; frequently we will refer to a (closed) subset $C \subset X$ as a predicate $C : X \to [0,1]$, and vice versa. The advantage of speaking of subsets in terms of predicates is that we can speak of predicates in terms of a given signature without needing to specify a specific structure for that signature. Whenever we have some signature S with a predicate symbol C and some S-theory Σ containing the s.a.p. axioms for C, we call C a *subset predicate* (with respect to Σ).

Returning to the issue of formalizing convexity, we might now ask ourselves how to deal with the implication in its definition: we need a continuous analogue of the expression

$$\forall x, \forall y\, (x \in C \wedge y \in C) \rightarrow L(x, y, t) \in C. \tag{3.2}$$

While Proposition 3.13 gives us a way of finding predicates $C : X \rightarrow [0, 1]$ so that we can transform (3.2) into

$$\forall x, \forall y\ (\max(C(x), C(y)) = 0) \rightarrow C(L(x, y, t)) = 0, \tag{3.3}$$

there is no completely satisfactory way to deal with the implication. For example, one might require some (uniformly continuous, monotonically increasing) "modulus of convexity" $u : [0, 1] \rightarrow [0, 1]$ satisfying $u(0) = 0$ and translate (3.3) as

$$\sup_x \sup_y\, (C(L(x, y, t)) \dot{-} u(\max(C(x), C(y)))) = 0 \tag{3.4}$$

which is certainly a formula expressible in continuous logic. The problem is that (3.4) is an a priori stronger condition than convexity, because even for pairs of points outside of the subset C it requires the potential failure of convexity between those points to be "no worse" than their failure to be inside of C, in the sense specified by the modulus u.

Indeed, from this we see that a faithful translation of convexity would actually be (3.4) with u the *discontinuous* function given by $u(0) = 0$ and $u(x) = 1$ otherwise. But we cannot allow discontinuous connectives (which is what such a u would be), because doing so would mean that ultraproducts (modified or not) of models of some theory Σ would no longer necessarily themselves be models of Σ.

Another solution would be to work with multiple sorts and regard C as its own space with its own linear structure L_C alongside the space X with its linear structure L_X, and having an inclusion map $i : C \hookrightarrow X$. We can then say that C is convex when the linear structure of C coincides with that of X, i.e. $i \circ L_C = L_X \circ (i \times i \times 1_{[0,1]})$ since by default C must be closed under its own linear structure.

A similar approach is to just regard the subset C as the whole space, and forget about X and questions about the convexity of C; this is unproblematic if the behavior/properties of the space X outside of the subset C happen to be unimportant. This is the approach we will take in the applications later in this paper.

We take the last part of this section to address a minor subtlety resulting from our change of setting from metric spaces to pseudometric spaces:

Definition 3.14. Let (X, d) be a pseudometric space and $T : X \to X$ some map. We say that p is a *fixed point* of T when $d(p, Tp) = 0$.

That is, "fixed point" is understood to mean "a point that is mapped by T to some (possibly distinct) point at distance 0", a necessary weakening of the usual definition of "fixed point" since we are working with pseudometrics instead of metrics. In general, this is an ill-behaved notion in the context of pseudometric spaces and arbitrary maps, since then we might have that $d(x, Tx) = 0$ but that possibly $d(x, T^n x) > 0$ for some $n > 1$. However, in our applications we will see that the very conditions that guarantee the existence of a fixed point in the above sense also ensure that $d(x, Tx) = 0$ implies $d(x, T^n x) = 0$ for all $n \geq 1$.

This is not a coincidence; fixed point results are commonly obtained through metric arguments that show that the distances between successive terms in a given kind of sequence converge to 0, and in the presence of the conditions that enable such arguments, we should reasonably expect that $d(x, Tx) = 0$ implies $d(x, T^n x) = 0$.

Remark 3.15. Note that the above definition of fixed point does not affect the usual definition of a convergent sequence, which is already defined only in terms of the values of the pseudometric; a sequence converges to a point p if and only if it converges to any other point q with $d(p, q) = 0$, i.e. convergence of a sequence only matters "up to distance 0". In a complete space this is of course equivalent to the sequence being Cauchy.

3.2 The modified ultraproduct

We assume the setting of "optionally continuous logic" (OCL) described in Chapter 3.

Let S be a signature of OCL. Let I be some index set and $\mathcal{X}_i$ some collection, indexed by I, of S-structures, each with underlying pseudometric space (X, d). As usual we assume that our spaces be bounded in diameter by 1 (if one wants to work with unbounded spaces, one can use sorts to stratify the spaces into bounded spaces of increasing diameter with inclusion maps between them).

Let $\mathcal{F}$ be some ultrafilter on I. We take $X = (\prod\limits_{i \in I} X_i)/\sim_{\mathcal{F}}$ where we declare that $(x_i) \sim_{\mathcal{F}} (y_i)$ when $\{i \in I | x_i = y_i\} \in \mathcal{F}$. We denote by $(x_i)_{\mathcal{F}}$ the equivalence class in X represented by (x_i). We still define the pseudometric d on X to be the same as in the usual continuous logic, i.e. $d((x_i)_{\mathcal{F}}, (y_i)_{\mathcal{F}}) = \lim\limits_{i,\mathcal{F}} d_i(x_i, y_i)$.

The rest follows naturally: if in addition we are also given (possibly discontinuous) functions $f_i : X_i \to Y_i$, then it is clear what $f : X \to Y$ should be, and that it is well-defined. If we are given predicates $R_i : X_i \to [0, 1]$, then we define $R : X \to [0, 1]$ as $R((x_i)_{\mathcal{F}}) = \lim\limits_{i,\mathcal{F}} R_i(x_i)$.

Now let us assume the setting of geodesic logic, so that S is now a geodesic signature. The above construction of the ultraproduct in OCL carries over wholesale to this setting; so now we need only to treat the linear structures L_i on X_i. If we treat the L_i like (the interpretations of) any other function in our signature, we see that the ultraproduct of the L_i gives a map $X \times X \times ([0,1]^I)/\sim_{\mathcal{F}} \to X$. That is, the ultraproduct of the linear structures L_i does *not* specify a linear structure on X, and so we see that linear structures must be treated differently under the ultraproduct.

Therefore, given a family L_i of linear structures corresponding to a family of S-structures $\mathcal{X}_i$, we do *not* define (the interpretation of) L to be the function

$$\hat{L} : X \times X \times ([0,1]^I)/\sim_{\mathcal{F}} \to X$$

that arises as the ultraproduct of the linear structures L_i considered as functions, but rather its restriction across the natural embedding $i_{\mathcal{F}} : [0,1] \hookrightarrow ([0,1]^I)/\sim_{\mathcal{F}}$. That is, $L = \hat{L} \circ (1_X, 1_X, i_{\mathcal{F}}) : X \times X \times [0,1] \to X$. This way, an ultraproduct of spaces equipped with linear structures itself has a linear structure.

It is easy to see that, defined in this way, for each $t \in [0,1]$ the t-value L_t of the ultraproduct linear structure L is exactly the ultraproduct of the t-values L_t^i treated as binary functions.

Theorem 2.6 - and therefore also Theorem 2.7 - is still valid in this setting (with the same proof). It is then immediate that the ultraproduct defined in this way is of hyperbolic type if all of its factors are.

Chapter 4

Fixed point results in analysis

The definitions and proofs in this section can be found in [9], [34] in the context of Banach spaces, but we will present them here in the context of spaces of hyperbolic type, where in many cases no alteration is required, and in some cases only slight adjustments to definitions/proofs are necessary.

Going forward, unless otherwise stated, we denote by (X, d, L) a pseudometric space of hyperbolic type, by $C \subset X$ a (nonempty) subset of X, and by $T : C \to X$ a function from C into X with a priori no special properties (such as continuity).

4.1 Condition ($\mathbf{C}_\lambda$) and Condition ($\mathbf{E}_\mu$)

Definition 4.1. A sequence $\{x_n\}$ of points of C is said to be an *almost fixed point sequence* (or *a.f.p.s.*, for short) for T when $\{x_n\}$ satisfies $d(x_n, Tx_n) \to 0$.

Definition 4.2. ([9])

Given $\mu \geq 1$, we say that T *satisfies condition (E_μ)* when $\forall x, y \in C$ we have that $d(x, Ty) \leq \mu\, d(x, Tx) + d(x, y)$.

We have the following obvious consequence of Definition 4.2:

Proposition 4.3. ([9])

If $T : C \to X$ satisfies condition (E_μ), and if $x_0 \in C$ is a fixed point of T, then for every $x \in C$ we have that $d(x_0, Tx) \leq d(x_0, x)$.

The importance of condition (E_μ) is that, in the presence of compactness, it guarantees an equivalence between having a fixed point and having an a.f.p.s.:

Theorem 4.4. ([9])

If C is compact and $T : C \to X$ satisfies condition (E_μ), then T has a fixed point if and only if T admits an a.f.p.s.

Proof. Given an a.f.p.s. $\{x_n\}$, pick a subsequence $\{x_{n_k}\}$ converging to some $x \in C$. We have:

$$\forall k,\ d(x_{n_k}, Tx) \leq \mu\, d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x), \text{ and}$$
$$\forall k,\ d(x, Tx) \leq d(x_{n_k}, Tx) + d(x_{n_k}, x)$$

which together imply that $d(x, Tx) = 0$. $\square$

It turns out that if C and $T : C \to X$ are nicer (but T still possibly discontinuous), we can actually guarantee the existence of an a.f.p.s. for T:

Definition 4.5. [34]

Given $\lambda \in [0, 1)$, we say that $T : C \to X$ *satisfies condition (C_λ)* when $\forall x, y \in C$, we have that $\lambda\, d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$.

From the above definition, we see that nonexpansive mappings T are exactly the ones which satisfy condition (C$_\lambda$) for $\lambda = 0$. Also, note that if $\lambda \leq \lambda'$, condition (C$_\lambda$) implies condition (C$_{\lambda'}$). Therefore, in what follows, we will assume w.l.o.g. that $\lambda > 0$.

Theorem 4.6. ([9], [11])

Let C be a bounded convex subset of X, with $T : C \to C$ satisfying condition (C_λ). Then there exists an a.f.p.s. for T, namely:

Let x_1 be any point in C, and let $x_{n+1} = L(x_n, Tx_n, \lambda)$. Then the sequence $\{x_n\}$ is an a.f.p.s. for T.

The sequence defined above is called a *Mann iteration* for T (starting at x_1). To emphasize the role of λ, we will call it a λ-*Mann iteration* for T (starting at x_1).

The key point to proving this is the following useful lemma, which was originally proven by [11] for metric spaces of hyperbolic type and then applied to the case of Banach spaces in [34] - and which we now observe actually applies to the more general case of pseudometric spaces of hyperbolic type:

Lemma 4.7. ([11], [34])

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a pseudometric space X of hyperbolic type, and let $\lambda \in (0, 1)$, such that $x_{n+1} = L(x_n, y_n, \lambda)$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all n. Then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

The original proof of Lemma 4.7 applied to metric spaces of hyperbolic type, but the unmodified proof also applies to pseudometric spaces of hyperbolic type. We give the proof, copied essentially verbatim from [11], in the Appendix so that the reader may verify this assertion for themselves.

The point is that once we are given the linear structure L on our pseudometric space X which satisfies the hyperbolicity condition, the proof, which is a lengthy string of inequalities, follows entirely mechanically. This is not to say that the proof does not make use of clever manipulations - only that, once the value λ and the sequences $\{x_n\}$ and $\{y_n\}$ are specified as in the statement of the Lemma, the proof depends purely on algebraic manipulation of inequalities involving those objects that result from X being a pseudometric space of hyperbolic type, and not, say, any argument that requires points at distance 0 to be the same point.

Proof of Theorem 4.6. If we can show that $d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1})$ for each n, then the rest is immediate from Lemma 4.7 by letting $\{y_n\} = \{Tx_n\}$.

Let $n \geq 1$. By construction we have $\lambda\, d(x_n, Tx_n) = d(x_n, x_{n+1})$, so by condition ($\mathrm{C}_\lambda$) we have that $d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1})$. $\square$

We have the following fixed point result as a corollary to Theorem 4.4 and Theorem 4.6:

Corollary 4.8. ([9])

If C is a compact, convex subset of X, and $T : C \to C$ satisfies condition (C_λ) for some $\lambda \in (0, 1)$ and condition (E_μ) for some $\mu \geq 1$, then T has a fixed point.

We have so far looked at properties of maps T satisfying condition (C_λ) and condition (E_μ). To summarize, Theorem 4.6 shows that condition (C_λ) along with certain conditions on the domain/codomain of the map T guarantees a sequence which is nice in some asymptotic sense (i.e. is an a.f.p.s.), and then Theorem 4.4 along with compactness of the domain guarantees a fixed point of T to which a subsequence of this a.f.p.s. converges.

Considering that Theorem 4.6 obtains this a.f.p.s. as a λ-Mann iteration for T for some $\lambda \in (0,1)$, we see that in fact that the entire sequence must converge:

Proposition 4.9. *Let $C \subset X$ and $T : C \to C$ fulfill the conditions of Theorem 4.4 and Theorem 4.6 (with some value $\lambda \in (0,1)$).*

Let $\{x_n\}$ be a λ-Mann iteration for T, as given in Theorem 4.6. Then $\{x_n\}$ converges to the fixed point x guaranteed by Theorem 4.4.

Proof. We have that $x_{n+1} = L(x_n, Tx_n, \lambda)$. Then by hyperbolicity we have

$$d(x, x_{n+1}) \leq (1-\lambda)\, d(x, x_n) + \lambda\, d(x, Tx_n).$$

By Proposition 4.3 we have that $d(x, Tx_n) \leq d(x, x_n)$ so that $d(x, x_{n+1}) \leq d(x, x_n)$. Since by Theorem 4.4 a subsequence of $\{x_n\}$ converges to x, we must have that $\{x_n\}$ itself must converge to x.

□

4.2 A generalization of Condition (C_λ)

We briefly look at a related property of maps $T : C \to C$ on $C \subset X$, which will serve to illuminate further discussion of condition (C_λ):

Definition 4.10. ([15])

For $C \subset X$, a map $T : C \to C$ is *directionally nonexpansive* if, for all $\lambda \in [0,1]$ and all $x \in C$, we have that $d(Tx, TL(x, Tx, \lambda)) \leq \lambda\, d(x, Tx)$.

The moral content of the above definition is that a directionally nonexpansive map T is one that is nonexpansive on the line segment between x and Tx.

Furthermore in [15] Kirk cites [8] in defining *asymptotic regularity* of $f : C \to C$ as the condition that for all $x \in C$, $\lim_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0$. With this, he proves the following theorem which bears striking resemblance to Theorem 4.6:

Theorem 4.11. ([15])

Let C be a bounded convex subset of X, and let $T : C \to C$ be directionally nonexpansive. Fix $\lambda \in (0,1)$, and define $f_T : C \to C$ by $f_T(x) = L(x, Tx, \lambda)$. Then f_T is asymptotically regular, and this convergence is uniform with respect to the choice of x and T.

Note that this theorem does not give uniformity with respect to C (however, such uniformity, along with even stronger results about the rate of convergence, is obtained in [21]). Given $x_1 \in C$ and f_T as above, the sequence $\{f_T^n(x_1)\}$ is precisely the λ-Mann iteration for T starting at x_1. Furthermore, asymptotic regularity of f_T is exactly equivalent to the λ-Mann iteration $\{x_n\} = \{f_T^n(x_1)\}$ being an a.f.p.s. for every starting point x_1, since $d(x_n, x_{n+1}) = \lambda d(x_n, Tx_n)$.

Remark 4.12. The connection just described actually runs deeper. In the proof of Theorem 4.6, we take a λ-Mann iteration $\{x_n\}$ and use condition (C_λ) to conclude, since $\lambda\, d(x_n, Tx_n) = d(x_n, x_{n+1})$, that $d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1})$; then we simply apply Lemma 4.7 to get that $\{x_n\}$ is an a.f.p.s.

Since by construction we have that $x_{n+1} = L(x_n, Tx_n, \lambda)$, it suffices to forget about condition (C_λ) and simply require that $d(Tx, TL(x, Tx, \lambda)) \leq d(x, L(x, Tx, \lambda)) = \lambda\, d(x, Tx)$ for all $x \in C$ - call this *condition (D_λ)* - to ensure that the proof of Theorem 4.6 nevertheless goes through.

Although condition (D_λ) simply assumes the conclusion of condition (C_λ) in the case of e.g. λ-Mann iterations, condition (D_λ) is actually *weaker* than condition (C_λ). Indeed, for any $x \in C$ we always have that $\lambda\, d(x, Tx) = d(x, L(x, Tx, \lambda))$ so that having condition (C_λ) would imply that condition (D_λ) holds.

In light of Definition 4.10, we see that condition (D_λ) is also a weak form of directional nonexpansiveness. Indeed, $T : C \to C$ is directionally nonexpansive precisely when it satisfies condition (D_λ) for every $\lambda \in [0, 1]$.

We formalize this discussion as follows:

Definition 4.13. Given $\lambda \in [0, 1]$, we say that $T : C \to X$ satisfies *condition (D_λ)* when $\forall x \in C$, we have that $d(Tx, TL(x, Tx, \lambda)) \leq \lambda\, d(x, Tx)$.

Proposition 4.14.

(a) Given $\lambda \in [0, 1)$, if $T : C \to X$ satisfies condition (C_λ) then it satisfies condition (D_λ). Furthermore, the conclusion of Theorem 4.6 remains true if we require that $T : C \to C$ satisfy condition (D_λ) instead of condition (C_λ) (with the other conditions unchanged).

(b) $T : C \to C$ is directionally nonexpansive if and only if it satisfies condition (D_λ) for every $\lambda \in [0, 1]$.

We note that any other result that we mention that refers to the conclusion of Theorem 4.6 also remains true if we replace condition (C_λ) with condition (D_λ).

One point of caution, however, is that for $\lambda < \lambda'$ we do not necessarily have that condition (D_λ) implies condition ($\mathrm{D}_{\lambda'}$). We will nevertheless restrict ourselves to the cases where $\lambda \in (0, 1)$, since those are the cases of interest, i.e. the ones to which Theorem 4.6 applies.

Chapter 5

Closure under ultraproducts

Thus far, we have reformulated the definitions and results of [9], [11], [34] (and [15] to a lesser extent), which were given in terms of metric/Banach spaces, in terms of pseudometric spaces of hyperbolic type.

We must now formalize all of this in the language of geodesic logic, which will allow us to use (the geodesic analogue of) Theorem 2.7, and thus obtain the existence of a uniform bound on the rate of metastable convergence in the results of e.g. Theorem 4.4 and Theorem 4.6.

Since the argument of Theorem 2.7 crucially requires passing to the ultraproduct, our approach to formalizing the objects and properties discussed in Chapter 4 in the framework of geodesic logic will follow the guiding principle that said properties should be preserved under taking ultraproducts.

As with [5], whenever we speak of ultrafilters/ultraproducts henceforth, we will assume that the set over which we are taking the ultrafilter is $\mathbb{N}$, and that the ultrafilter is non-principal.

We have already seen, via the expression (3.1), how to formalize the property of a space being of hyperbolic type.

5.1 Compactness

We must currently address two issues which will turn out to have the same solution. The first is that the functions $T : C \to C$ that we are interested in are only partially defined on X (i.e. they are not functions $T : X \to X$), and so without further modification cannot be considered honest interpretations of function symbols. The second issue is that, if we are to have a class, closed under taking ultraproducts, of (structures on) spaces X specifying a special subset $C \subset X$ of each space, each of which is required to be convex, then we must formulate some notion of convexity that the subsets C must obey in a manner which is somehow uniform across the members of the class.

The solution is simply to note that the results we are interested in (e.g. Theorem 4.4 and Theorem 4.6) and their proofs concern themselves only with the features of the subset $C \subset X$. Since we will require C to be convex anyway (so that the linear structure on X restricts to give a linear structure on C), we can simply regard C as the entire space. Therefore, in what follows, when we refer to structures $\mathcal{X}$ and the spaces X associated with them, it should be understood that we intend them to play the role of the *subsets* $C \subset X$ from the results of Chapter 4. In this way we get convexity automatically from simply having a linear structure.

Now we would like our ultraproduct to inherit properties such as compactness (which Theorem 4.4 requires) from its factors. More precisely, if we have a family X_i of compact spaces then we would like the ultraproduct X to inherit those properties. This is unfortunately not the case in general. However, in the case of pseudometric spaces, compactness is equivalent to being complete and totally bounded.

We have already started out assuming that our "base" spaces will be complete. That their ultraproducts are again complete is simple: ultraproducts are ω_1-saturated, which among other things guarantees Cauchy completeness [12].

For total boundedness, we borrow Kohlenbach's idea [17] of specifying a modulus of total boundedness, which is a way to ensure that a family of structures with that modulus is totally bounded in some uniform way. We also give an alternative notion of total boundedness which is equivalent (as Proposotion 5.2 will show) yet easier to work with.

Definition 5.1. Let (X, d) be a pseudometric space.

(a) We say that X is *totally bounded* when, for every $K \in \mathbb{N}$, there is some $\alpha(K) \in \mathbb{N}$ such that there exist points $x_0, \ldots, x_{\alpha(K)}$ such that $\min(d(x, x_0), \ldots, d(x, x_{\alpha(K)})) < \frac{1}{K+1}$.

This function $\alpha : \mathbb{N} \to \mathbb{N}$ is called a *modulus of total boundedness* for X.

(b) We say that X is *approximately totally bounded* when, for every $k \in \mathbb{N}$, there is some $\beta(k) \in \mathbb{N}$ such that the following holds:

$$\inf_{x_0} \cdots \inf_{x_{\beta(k)}} \sup_{x} (\min(d(x, x_0), \ldots, d(x, x_{\beta(k)}))) \leq \frac{1}{k+1} \tag{5.1}$$

This function $\beta : \mathbb{N} \to \mathbb{N}$ is called a *modulus of approximate total boundedness* for X.

Note that the condition given by (5.1) in Definition 5.1 (b) can be restated as follows:

$$\inf_{x_0} \cdots \inf_{x_{\beta(k)}} \sup_{x} (\min(d(x, x_0), \ldots, d(x, x_{\beta(k)})) \dot{-} \frac{1}{k+1}) = 0. \tag{5.1$'$}$$

So a pseudometric space X is totally bounded if and only if it has some modulus of total boundedness, and approximately totally bounded if and only if it has some modulus of approximate total boundedness. We now show that these two conditions are actually equivalent:

Proposition 5.2. *Let (X, d) be a pseudometric space. The following are equivalent:*

(a) X is totally bounded.

(b) X is approximately totally bounded.

Proof. It is clear that X being totally bounded implies that X is approximately totally bounded: if α is a modulus of total boundedness for X, $\beta = \alpha$ is a modulus of approximate total boundedness for X.

Conversely, let $\beta : \mathbb{N} \to \mathbb{N}$ be a modulus of approximate total boundedness. We need to produce a function $\alpha : \mathbb{N} \to \mathbb{N}$ such that for each $K \in \mathbb{N}$, there exist finitely many points $x_1, \ldots, x_{\alpha(K)}$ such that for each $x \in X$, we have that $\min(d(x, x_0), \ldots, d(x, x_{\alpha(K)})) < \frac{1}{K+1}$. So given $K \in \mathbb{N}$, choose $k \in \mathbb{N}$ to be such that $\frac{2}{k+1} < \frac{1}{K+1}$. Then by assumption there exist points $x_0, \ldots, x_{\beta(k)}$ such that for each x, we have that

$$\min(d(x, x_0), \ldots, d(x, x_{\beta(k)})) < \frac{2}{k+1} < \frac{1}{K+1}.$$

Then $\alpha : \mathbb{N} \to \mathbb{N}$ defined by this assignment $K \mapsto k \mapsto \beta(k)$ is a modulus of total boundedness for X.

$\square$

The advantage of working with approximate total boundedness is that, as $(5.1')$ shows, the notion of approximate total boundedness is easily formalized in our logic. In fact, since it neither requires a linear structure nor refers to any discontinuous functions, it is also formalizable in the usual continuous logic - but here we will restrict our discussions to geodesic logic, which is the one we need to use for our applications.

Definition 5.3. Let S be a geodesic signature, and $\mathcal{X}$ an S-structure.

We say that $\mathcal{X}$ is *totally bounded* when there is a function $\beta : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\mathcal{X}$ satisfies the S-condition

$$\inf_{x_0} \cdots \inf_{x_{\beta(k)}} \sup_{x} (\min(d(x, x_0), \ldots, d(x, x_{\beta(k)})) \dot{-} \frac{1}{k+1}) = 0.$$

We call this β a *modulus of total boundedness* for $\mathcal{X}$.

By Proposition 5.2, $\mathcal{X}$ is totally bounded in the above sense if and only if the underlying space X is totally bounded in the sense of Definition 5.1.

It is clear from Definition 5.3 that if $\beta : \mathbb{N} \to \mathbb{N}$ is a modulus of total boundedness for a family $\mathcal{X}_i$ of S-structures, then β is a modulus of total boundedness for the ultraproduct $\mathcal{X}$ of the $\mathcal{X}_i$.

Remark 5.4. Note that even in the absence of total boundedness, any family of S-structures for a given signature S automatically shares a bound on the diameters of their underlying spaces, by Definition 3.6.

5.2 Formalization into geodesic logic

So far we have seen how to incorporate notions of hyperbolic type, convexity, and compactness into the framework of our logic. It remains to express condition (E_μ) and condition

(D_λ) in geodesic logic. It is here that the importance of choosing condition (D_λ) over condition (C_λ) becomes clear; while Proposition 4.14 justifies why we *can* do so, the reason why we *want* to is that condition (D_λ) is much simpler to formalize, because it does not contain any implications (refer to the discussion occurring after Definition 3.12 for why implications are problematic in our logic).

Definition 5.5. Let S be a geodesic signature with a unary function symbol T, and let $\mathcal{X}$ be an S-structure.

(a) Let $\mu \geq 1$. We say that $\mathcal{X}$ *satisfies condition (E_μ)* when $\mathcal{X}$ satisfies the S-condition

$$\sup_x \sup_y \left((d(x, Ty) \dot{-} \mu\, d(x, Tx)) \dot{-} d(x, y) \right) = 0.$$

(b) Let $\lambda \in (0, 1)$. We say that $\mathcal{X}$ *satisfies condition (D_λ)* when $\mathcal{X}$ satisfies the S-condition

$$\sup_x \left(d(Tx, TL_\lambda(x, Tx)) \dot{-} \lambda\, d(x, Tx) \right) = 0.$$

Letting X be the underlying space of $\mathcal{X}$ and $T : X \to X$ the interpretation of the function symbol T, it is straightforward to see that $\mathcal{X}$ satisfies Definition 5.5 (a) if and only if $T : X \to X$ satisfies condition (E_μ) in the sense of Definition 4.2, since

$$\begin{aligned} & \forall x, y \in X,\ d(x, Ty) \leq \mu\, d(x, Tx) + d(x, y) \\ \iff \quad & \sup_x \sup_y \left((d(x, Ty) \dot{-} \mu\, d(x, Tx)) \dot{-} d(x, y) \right) = 0. \end{aligned}$$

Similarly, we see that $\mathcal{X}$ satisfies Definition 5.5 (b) if and only if $T : X \to X$ satisfies condition (D_λ) in the sense of Definition 4.13, since

$$\begin{aligned} & \forall x \in X,\ d(Tx, TL(x, Tx, \lambda)) \leq \lambda\, d(x, Tx) \\ \iff \quad & \sup_x \left(d(Tx, TL(x, Tx, \lambda)) \dot{-} \lambda\, d(x, Tx) \right) = 0. \end{aligned}$$

From this we see that an ultraproduct of structures satisfying condition (D_λ) (resp. condition (E_μ)) itself satisfies condition (D_λ) (resp. condition (E_μ)).

5.3 The main results

We are now ready to apply the Avigad-Iovino approach to Theorem 4.6.

Theorem 5.6. *Let S be a geodesic signature with a unary function symbol T and a constant symbol x_1.*

Let $\lambda \in (0, 1)$ be given, and let $\mathcal{C}$ be the class of S-structures of hyperbolic type satisfying condition (D_λ).

For each $\mathcal{X} \in \mathcal{C}$, let $\{x_n\}$ be the sequence defined by $x_{n+1} = L_\lambda(x_n, Tx_n)$. Then we have the following:

(a) Letting $d_n = d(x_n, x_{n+1})$, we have that $\lim_{n\to\infty} d_n = 0$. Equivalently, $\{x_n\}$ is an a.f.p.s. for T.

(b) Given any function $F : \mathbb{N} \to \mathbb{N}$, there is a bound on the rate of metastability of the above convergence with respect to F, which is uniform in $\mathcal{X} \in \mathcal{C}$.

Remark 5.7. Recall from Definition 1.2 that, given $F : \mathbb{N} \to \mathbb{N}$, a *bound on the rate of metastability* for a sequence $\{d_n\}$ is a function $b_F : \mathbb{R}_{>0} \to \mathbb{N}$ such that for each $\epsilon > 0$ there exists some $n \leq b_F(\epsilon)$ such that for all $i, j \in [n, F(n)]$, we have that $d(d_i, d_j) < \epsilon$. (In the specific case of Theorem 5.6, $d(d_i, d_j) = |d_i - d_j|$.)

Theorem 5.6 is a simultaneous generalization of Theorem 4.6 and Theorem 4.11, since condition (D_λ) is a weaker condition than both condition (C_λ) (used in Theorem 4.6) and directional nonexpansiveness (used in Theorem 4.11).

Furthermore, since the data of each structure $\mathcal{X}$ includes not only the space X but also the function $T : X \to X$ as well as the choice of starting point $x_1 \in X$, Theorem 5.6 (b) guarantees a bound on the "metastable asymptotic regularity" of the λ-Mann iterations that is uniform in X, functions $T : X \to X$, and starting points $x_1 \in X$.

Proving Theorem 5.6 will involve the following lemma which is a variant of Theorem 2.7:

Lemma 5.8. *Let S be a geodesic signature, and let $\{t_n\}$ be a sequence of S-terms.*

Let $\mathcal{C}$ a class of S-structures. For each $\mathcal{X} \in \mathcal{C}$, let $\{x_n\}$ denote the interpretation in $\mathcal{X}$ of the sequence $\{t_n\}$, and let $d_n = d(x_n, x_{n+1})$.

Finally, let $\mathcal{F}$ be an ultrafilter. Then the following are equivalent:

(a) For every $\epsilon > 0$ and every $F : \mathbb{N} \to \mathbb{N}$, there is some $b \geq 1$ such that the following holds: for every $\mathcal{X}$ in $\mathcal{C}$, there is an $n \leq b$ such that $d_i < \epsilon$ for every $i \in [n, F(n)]$.

(b) For any sequence $\mathcal{X}_k$ of elements of $\mathcal{C}$, let $\mathcal{X}$ be their $\mathcal{F}$-ultraproduct. Then for every $\epsilon > 0$ and every $F : \mathbb{N} \to \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $d_i < \epsilon$ for every $i \in [n, F(n)]$.

Proof. The proof is essentially the same as for Theorem 2.7.

(a) $\Rightarrow$ (b): For any fixed $\frac{1}{2}\epsilon > 0$ and any fixed $F : \mathbb{N} \to \mathbb{N}$, there is a $b \geq 1$ such that every member of $\mathcal{C}$ satisfies the condition

$$\min_{n \leq b} \left(\max_{i \in [n, F(n)]} (d_i \dot{-} \tfrac{1}{2}\epsilon) \right) = 0.$$

Thus any ultraproduct of members of $\mathcal{C}$ must again be a model of this sentence.

(b) $\Rightarrow$ (a): Assume that (a) fails. That is, for some $\epsilon > 0$ and some $F : \mathbb{N} \to \mathbb{N}$, for each $k \in \mathbb{N}$ there is a member $\mathcal{X}_k$ of $\mathcal{C}$ such that for every $n \leq k$ and for some $i \in [n, F(n)]$, we have $d_i^k \geq \epsilon$. Let $\mathcal{X}$ be the $\mathcal{F}$-ultraproduct of the sequence $\mathcal{X}_k$ thus obtained.

Given any n, since there are cofinitely many $k \geq n$, it is also true for cofinitely many k that there is some $i \in [n, F(n)]$ with $d_i^k \geq \epsilon$. It follows that there is some specific $i \in [n, F(n)]$ such that $d_i^k \geq \epsilon$ for $\mathcal{F}$-many k, so that $d_i = \lim_{k,\mathcal{F}} d_i^k \geq \epsilon$ for that i. Since n was arbitrary, we see that (b) fails.

$\square$

Proof of Theorem 5.6.

(a): For each $\mathcal{X} \in \mathcal{C}$, $\{x_n\}$ is the λ-Mann iteration for T starting at x_1. By having specified a geodesic signature S we automatically have that the underlying space X is bounded and convex with respect to the linear structure. Since $\mathcal{X}$ is of hyperbolic type and $T : X \to X$ satisfies condition (D_λ), we can use Proposition 4.14 to apply Theorem 4.6 and conclude that $\{x_n\}$ is an a.f.p.s. for T. And since $d_n = \lambda\, d(x_n, Tx_n)$, we equivalently have that $\lim_{n\to\infty} d_n = 0$.

(b): For each $\mathcal{X} \in \mathcal{C}$, the sequence $\{x_n\}$ is the interpretation in $\mathcal{X}$ of the sequence of S-terms $\{t_n\}$ where $t_1 = x_1$ and $t_{n+1} = L_\lambda(t_n, Tt_n)$.

Furthermore, given an ultrafilter $\mathcal{F}$, for any sequence $\mathcal{X}_k$ of elements of $\mathcal{C}$, their $\mathcal{F}$-

ultraproduct $\mathcal{X}$ is again an S-structure (so bounded and convex) of hyperbolic type satisfying condition (D_λ), so that $\lim_{n\to\infty} d_n = 0$. By Proposition 1.4, we see that part (b) of Lemma 5.8 is satisfied, so that we have part (a) of that lemma as well, which gives us Theorem 5.6 (b).

□

Now that we have obtained a uniform version of Theorem 4.6, we now consider the case where we also have compactness (total boundedness) and condition (E_μ):

Theorem 5.9. *Let S be a geodesic signature with a unary function symbol T and a constant symbol x_1.*

Let $\lambda \in (0,1)$, $\mu \geq 1$, and $\beta : \mathbb{N} \to \mathbb{N}$ be given.

Let $\mathcal{C}$ be the class of S-structures of hyperbolic type which have β as a modulus of total boundedness, and which satisfy condition (D_λ) and condition (E_μ).

Finally, for each $\mathcal{X} \in \mathcal{C}$, let $\{x_n\}$ be the sequence defined by $x_{n+1} = L_\lambda(x_n, Tx_n)$. Then we have the following:

(a) T has a fixed point toward which $\{x_n\}$ converges.

(b) For each $F : \mathbb{N} \to \mathbb{N}$, there is a bound on the rate of metastability for the above convergence which is uniform in $\mathcal{X} \in \mathcal{C}$.

Proof. (a): For each $\mathcal{X} \in \mathcal{C}$, we have that $\{x_n\}$ is the λ-Mann iteration for T starting at x_1. The underlying space X is convex and compact, $\mathcal{X}$ is of hyperbolic type, and $T : X \to X$ satisfies condition (D_λ) and condition (E_μ), so $\{x_n\}$ is an a.f.p.s. for T, which then converges to a fixed point x by Theorem 4.4 and Proposition 4.9.

(b): As in the proof of Theorem 5.6 (b), for each $\mathcal{X}$ the sequence $\{x_n\}$ is the interpretation of the sequence $\{t_n\}$ of S-terms where $t_1 = x_1$ and $t_{n+1} = L_\lambda(t_n, Tt_n)$.

All of the relevant conditions - convexity, compactness, hyperbolic type, condition (D_λ), and condition (E_μ) - are preserved under ultraproducts. Thus given an ultrafilter $\mathcal{F}$ and any sequence $\mathcal{X}_k$ of elements of $\mathcal{C}$, the $\mathcal{F}$-ultraproduct $\mathcal{X}$ of the $\mathcal{X}_k$ is again in $\mathcal{C}$, so that the sequence $\{x_n\}$ associated with $\mathcal{X}$ converges. Thus by Theorem 2.7 (which, as we observed at the end of Section 3.2, is still valid for geodesic logic) we have Theorem 5.9 (b).

$\square$

Remark 5.10. As in Theorem 5.6, the bound on the rate of metastability guaranteed by the theorem above is uniform in the spaces X, functions $T : X \to X$, and choices of starting point $x_1 \in X$ for the λ-Mann iterations.

Appendix to Part I

Here we supply the proof of Lemma 4.7, to make it clear that the entire proof is valid, unmodified from [11], within the context of pseudometric spaces of hyperbolic type.

Lemma 4.7. ([11], [34])

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a pseudometric space X of hyperbolic type, and let $\lambda \in (0,1)$, such that $x_{n+1} = L(x_n, y_n, \lambda)$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all n. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Proof. The first claim is that, for all $i, n \in \mathbb{N}$:

$$(1 + n\lambda)\, d(x_i, y_i) \leq d(x_i, y_{i+n}) + (1-\lambda)^{-n}(d(x_i, y_i) - d(x_{i+n}, y_{i+n})) \tag{5.2}$$

If $n = 1$, then (5.2) simplifies to $(1+\lambda)\, d(x_i, y_i) \leq d(x_i, y_{i+1}) + \frac{1}{1-\lambda}(d(x_i, y_i) - d(x_{i+1}, y_{i+1}))$, which we can manipulate as follows:

$$\begin{aligned}
& (1+\lambda)\, d(x_i, y_i) \leq d(x_i, y_{i+1}) + \frac{1}{1-\lambda}(d(x_i, y_i) - d(x_{i+1}, y_{i+1})) \\
\iff \quad & d(x_{i+1}, y_{i+1}) \leq (1-\lambda) d(x_i, y_{i+1}) + \lambda^2\, d(x_i, y_i) \\
& \qquad = (1-\lambda) d(x_i, y_{i+1}) + \lambda\, d(x_i, x_{i+1})
\end{aligned}$$

where we have used the fact that $d(x_i, x_{i+1}) = \lambda\, d(x_i, y_i)$. But we know that $d(x_{i+1}, y_{i+1}) \leq (1-\lambda) d(x_i, y_{i+1}) + \lambda\, d(x_i, x_{i+1})$ by hyperbolicity, so (5.2) holds for $n = 1$ and all $i \in \mathbb{N}$.

So let us assume by induction that (5.2) is true for some n, and all i. By replacing i with $i+1$, we get:

$$\begin{aligned}(1+n\lambda)\, d(x_{i+1}, y_{i+1}) \leq\ & d(x_{i+1}, y_{i+n+1}) \\ & + (1-\lambda)^{-n}(d(x_{i+1}, y_{i+1}) - d(x_{i+n+1}, y_{i+n+1}))\end{aligned} \tag{5.3}$$

while from hyperbolicity and the rest of our assumptions we get:

$$\begin{aligned} d(x_{i+1}, y_{i+n+1}) &\leq (1-\lambda)\, d(x_i, y_{i+n+1}) + \lambda\, d(y_i, y_{i+n+1}) \\ &\leq (1-\lambda)\, d(x_i, y_{i+n+1}) + \lambda \sum_{k=i}^{i+n} d(y_k, y_{k+1}) \\ &\leq (1-\lambda)\, d(x_i, y_{i+n+1}) + \lambda \sum_{k=i}^{i+n} d(x_k, x_{k+1}) \end{aligned} \tag{5.4}$$

It is easy to check that our assumptions imply that $d(x_k, y_k) \leq d(x_{k+1}, y_{k+1})$ for all k. We use this fact and the aforementioned assumptions in the following derivation which combines (5.3) and (5.4):

$$
\begin{aligned}
d(x_i, y_{i+n+1}) &\geq (1-\lambda)^{-1} d(x_{i+1}, y_{i+n+1}) - \lambda(1-\lambda)^{-1} \sum_{k=i}^{i+n} d(x_k, x_{k+1}) \\
&\geq (1-\lambda)^{-1}(1+n\lambda)\, d(x_{i+1}, y_{i+1}) \\
&\qquad + (1-\lambda)^{-n-1} \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_{i+1}, y_{i+1})\right) \\
&\qquad - \lambda(1-\lambda)^{-1} \sum_{k=i}^{i+n} d(x_k, x_{k+1}) \\
&= (1-\lambda)^{-1}(1+n\lambda)\, d(x_{i+1}, y_{i+1}) \\
&\qquad + (1-\lambda)^{-n-1} \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_{i+1}, y_{i+1})\right) \\
&\qquad - \lambda^2(1-\lambda)^{-1} \sum_{k=i}^{i+n} d(x_k, y_k) \\
&\geq (1-\lambda)^{-1}(1+n\lambda)\, d(x_{i+1}, y_{i+1}) \\
&\qquad + (1-\lambda)^{-n-1} \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_{i+1}, y_{i+1})\right) \\
&\qquad - \lambda^2(1-\lambda)^{-1}(n+1)\, d(x_i, y_i) \\
&= (1-\lambda)^{-n-1} \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_i, y_i)\right) \\
&\qquad + (1-\lambda)^{-1} \big((1+n\lambda) - (1-\lambda)^{-n}\big)\ d(x_{i+1}, y_{i+1}) \\
&\qquad + \big((1-\lambda)^{-n-1} - \lambda^2(1-\lambda)^{-1}(n+1)\big)\ d(x_i, y_i)
\end{aligned}
$$

From e.g. the expression of each $\frac{1}{1-\lambda}$ as a power series, we have that $(1+n\lambda) \leq (1-\lambda)^{-n}$, so the last inequality above remains true when we replace $d(x_{i+1}, y_{i+1})$ by $d(x_i, y_i)$:

$$
\begin{aligned}
d(x_i, y_{i+n+1}) \geq (1-\lambda)^{-n-1} & \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_i, y_i)\right) \\
& + (1-\lambda)^{-1} \big((1+n\lambda) - \lambda^2(n+1)\big)\, d(x_i, y_i) \\
= (1-\lambda)^{-(n+1)} & \left(d(x_{i+n+1}, y_{i+n+1}) - d(x_i, y_i)\right) \\
& + (1+(n+1)\lambda)\, d(x_i, y_i)
\end{aligned}
$$

which completes the induction.

Having proven (5.2), we now show that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Assume otherwise, i.e. that there is some $r > 0$ such that $\lim_{n\to\infty} d(x_n, y_n) = r$. Let D denote a bound for the sequences $\{x_n\}$ and $\{y_n\}$.

We can pick $\epsilon > 0$ such that $\epsilon \exp\big((1-\lambda)^{-1}(r^{-1}D+1)\big) < r$.

Choose i so that for all $n \geq 1$, $d(x_i, y_i) - d(x_{i+n}, y_{i+n}) \leq \epsilon$, and choose N so that $\lambda r(N-1) \leq D \leq \lambda r N$. Then we have that $\lambda r N < D + r$ ($\Rightarrow N\lambda < r^{-1}D + 1$).

We also have:

$$
\begin{aligned}
(1-\lambda)^{-N} &= (1+(1-\lambda)^{-1})^N \\
&= \exp\big(N \log(1+\lambda(1-\lambda)^{-1})\big) \\
&\leq \exp\big(N\lambda(1-\lambda)^{-1}\big)
\end{aligned}
$$

So that we get the following contradiction:

$$
\begin{aligned}
D + r \leq (1 + N\lambda) r &\leq (1 + N\lambda) d(x_i, y_i) \\
&\leq d(x_i, y_{i+N}) + \epsilon \exp\left(N\lambda(1-\lambda)^{-1}\right) \\
&\leq D + \epsilon \exp\left((1-\lambda)^{-1}(r^{-1}D + 1)\right) \\
&< D + r
\end{aligned}
$$

$\square$

Part II

Continuous logic and enriched category theory

Chapter 6

Introduction

In his influential work [27], Lawvere demonstrated the conceptual power of enriching categories in categories other than that of **Set**; he exhibited a $\mathbb{R}$-enriched category as a (generalized) metric space while hypothesizing what an $\mathbb{R}$-valued logic look like, though neither quite arriving at the not-yet-existent continuous logic, nor capturing uniform continuity in his constructions (although he does so for Lipschitz continuity).

Now there is a well-developed interaction between classical model theory and category theory, which has its roots in Lawvere's thesis [26] on a categorical interpretation of algebraic theories. A mature form of this interaction can be found in [33], in which categorical interpretations of the features of classical first order logic are given. A yet more developed connection between model theory and category theory is found in the framework of abstract elementary classes and accessible categories (see e.g. [28], [32]).

The connection between *metric* model theory and category theory has developed in reverse chronological order: relatively recent work [13] introduces "metric abstract elementary classes" whose connection to accessible categories is developed in [29], and yet more recent work [1] elegantly describes the continuous analogue of the framework presented in [33]. In more detail, [1] defines "continuous syntactic categories", to be thought of as algebras of continuous functions into the interval $[0, 1]$ (i.e. continuous predicates) compatible with the algebra of continuous functions $[0, 1]^n \to [0, 1]$, and then gives conditions ensuring that such categories may be interpreted as (basically) honest metric spaces; in this setting the various continuous analogues of results in [33] are proven.

Our main project in the second part of this thesis fits into the above program by exhibiting a category of $\mathbb{R}$-enriched categories for which a suitable notion of "continuous" subobject naturally allows for an interpretation of continuous predicates and behaves like objects of the continuous syntactic categories above; in fact we are able to exhibit (the symmetric version of) $\mathbb{R}$ as a "continuous subobject classifier" in a precise sense analogous to the role of $\Omega = \{0, 1\}$ as a subobject classifier in **Set**. We arrive at these notions by first defining the framework to describe uniform continuity categorically, and then showing that this is sufficiently well-behaved as to ultimately allow for the above constructions and therefore an organic interpretation of continuous logic. Thus our present work should also be understood as an exploration of how the categorical structures required by [1] might arise "in the wild".

Chapter 7

Basics of categorical logic

We now recall some of the framework relevant to interpreting (classical, i.e. non-continuous) many-sorted first order logic in categories.

7.1 Syntax of first order logic

We begin with the basic syntactic notions of first order logic, gliding over technical subtleties when they are not relevant to our purposes and doing so would not result in confusion; the interested reader can find a more detailed treatment in [33].

Definition 7.1. A *signature* S consists of:

(a) A set $\mathcal{S}$ of *sort symbols* s_i, containing $*$ (the "terminal sort").

(b) A set $\mathcal{F}$ of *function symbols* f_j, such that for each $f \in \mathcal{F}$ we have data $(n, s_1, \ldots, s_n, s)$, where n is a natural number and each s_i is an element of $\mathcal{S}$. In this case we say that f is an n-ary function (symbol) of type $(s_1 \times \cdots \times s_n) \to s$, or of type $\left(\prod_{1 \leq i \leq n} s_i\right) \to s$. We may also write $f : (s_1 \times \cdots \times s_n) \to s$ or $f : \left(\prod_{1 \leq i \leq n} s_i\right) \to s$.

(c) A set $\mathcal{R}$ of *predicate symbols* R_k, such that for each $R \in \mathcal{R}$ we have data $(n, s_1, \ldots, s_n)$ where n is a natural number and each s_i is an element of $\mathcal{S}$. In this case we say that R is an n-ary predicate (symbol) of type $s_1 \times \cdots \times s_n$, or of type $\prod\limits_{1 \leq i \leq n} s_i$. We may also write $R \subset s_1 \times \cdots \times s_n$ or $R \subset \prod\limits_{1 \leq i \leq n} s_i$.

In specifying the type of a symbol, $\prod\limits_{\emptyset} s_i$ is to be understood as $*$. We will frequently refer to a 0-ary function symbol c of type $* \to s$ as a *constant symbol c of type s*.

We do not allow 0-ary predicate symbols R (we could, but they necessarily end up having trivial interpretations).

In addition to the symbols provided by our signature S (the *nonlogical* symbols), we also have the *logical symbols*: we have the equality symbol $=$, *connectives* $\{\wedge, \vee, \Rightarrow, \forall, \exists\}$, and for each $s \in \mathcal{S}$ we have an infinite set $\{x_i\}$ of *variables of type s*.

Now we briefly review the inductive constructions of terms and formulas given a signature S:

Definition 7.2. Let S be a signature.

(a) A *term* for S is given by the following inductive description:

  (i) Each variable x of type s is a term of type s, with free variable x.

  (ii) Each constant symbol c of type s is a term of type s, with no free variables.

  (iii) Let $t_1, \ldots, t_n$ be terms where t_k is of type s_k, and $f : \prod\limits_{1 \leq k \leq n} s_k \to s$. Then $f(t_1, \ldots, t_n)$ is a term of type s, with free variables given by the union over k of the free variables of each t_k (in particular we only count each distinct free variable once).

(b) Let $t_1, \ldots, t_n$ be terms where t_k is of type s_k, and $R \subset \prod_{1 \le k \le n} s_k$.

Then $R(t_1, \ldots, t_n)$ is an *atomic formula* with free variables given by the union of k of the free variables of each t_k. (The logical symbol $=$ is treated as a binary predicate symbol of type $s \times s$, where s can be any sort.)

(c) A *formula* for S is given by the following inductive description:

(i) Each atomic formula is a formula.

(ii) If ϕ and ψ are formulas then so are $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \Rightarrow \psi$, with free variables given by the union of the free variables of ϕ and ψ.

(iii) If ϕ is a formula and x is a free variable of ϕ then $\forall x\, \phi$ and $\exists x\, \phi$ are formulas with free variables equal to the free variables of ϕ omitting x.

(iv) If ϕ is a formula with no free variables, then ϕ is called a *sentence*.

An S-theory Σ is then just a set of sentences of S.

Briefly, a model of the language S is just an assignment of a set $[\![s]\!]$ to each sort symbol s, a set function $[\![f]\!] : (\prod_i [\![s_i]\!]) \to [\![s]\!]$ to each function symbol $f : \prod_i s_i \to s$, and a subset $[\![R]\!] \subset \prod_i [\![s_i]\!]$ to each predicate symbol $R \subset \prod_i s_i$. Such a collection of assignments then induces an assignment of a set function to each S-term and a subset to each S-formula. A model of an S-theory Σ is such an assignment for which the interpretation "makes each $\phi \in \Sigma$ true". We will describe all of this more precisely in a more general setting in the following sections.

7.2 Basic categorical notions

Before we can actually describe how to interpret the terms and formulas of a signature S into a category $\mathcal{C}$, we must first recall some basic constructions in category theory. For the most part we will not prove in detail (or sometimes at all) the results in this section, as they can be found in standard references such as [31]. We assume a rudimentary knowledge of categories, e.g. at the level of limits and adjunctions, for which the canonical reference is [30].

Definition 7.3. Let $\mathcal{C}$ be a category, and X an object of $\mathcal{C}$.

By a *subobject* ι of X we mean the isomorphism class (fixing X) of a monomorphism $\iota : A \hookrightarrow X$. If there is a commutative triangle

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, hook, "\iota"'] & & B \arrow[dl, hook, "\iota'"] \\ & X & \end{tikzcd}$$

then we say that $\iota \leq \iota'$ as subobjects of X.

By the nature of monomorphisms, the class of subobjects $\operatorname{Sub} X$ of a given object $X \in \mathcal{C}$ forms a poset, so in particular a category. If the category $\mathcal{C}$ has extra structure, then so does the poset $\operatorname{Sub} X$ (see Proposition 7.4 below). Note that 1_X is the terminal object of $\operatorname{Sub} X$, and if $\mathcal{C}$ has an initial object such that any morphism out of the initial object is monic then it is also an initial object in $\operatorname{Sub} X$. (Henceforth we make this assumption about every category $\mathcal{C}$ that we mention.)

Although objects of $\operatorname{Sub} X$ are, strictly speaking, isomorphism classes of monomorphisms $\iota : A \hookrightarrow X$, for convenience we will usually refer to a subobject ι by the domain of a representing monomorphism when there is no chance for confusion. Thus if $\iota : A \hookrightarrow X$ represents the subobject ι then we may refer to ι by A.

Proposition 7.4. *Let $\mathcal{C}$ be a category, and X an object of $\mathcal{C}$.*

(a) If $\mathcal{C}$ has finite limits, then $\mathrm{Sub}\, X$ has products (meets).

(b) If $\mathcal{C}$ also has finite colimits, then $\mathrm{Sub}\, X$ has coproducts (joins).

For more details, the reader is referred to [31]. Roughly, given $A \hookrightarrow X$ and $B \hookrightarrow X$ in $\mathrm{Sub}\, X$, the meet $A \wedge B$ of A and B in $\mathrm{Sub}\, X$ is given by the pullback

$$\begin{tikzcd}
A \wedge B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

while the join $A \vee B$ is induced from the pushout

$$\begin{tikzcd}
A \wedge B \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A \vee B
\end{tikzcd}$$

Thus for each $X \in \mathcal{C}$, $\mathrm{Sub}\, X$ has the structure of a lattice.

Example 7.5. When $\mathcal{C} = \mathbf{Set}$ and $X \in \mathbf{Set}$ then the powerset $\mathcal{P}(X)$ of X considered as a poset is isomorphic to $\mathrm{Sub}\, X$.

If we have $f : X \to Y$ in $\mathcal{C}$, then "pullback across f" gives a functor (i.e. an order-preserving poset map) $f^* : \mathrm{Sub}\, Y \to \mathrm{Sub}\, X$:

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

Example 7.6. When $\mathcal{C} = \mathbf{Set}$ with $f : X \to Y$ in **Set**, then the pullback functor $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ is given by $f^*A = f^{-1}(A) \subset X$ for a subset $A \subset Y$.

For categories $\mathcal{C}$ with sufficient structure, for each $f : X \to Y$ we have a left (resp. right) adjoint $\exists_f$ (resp. $\forall_f$) : $\mathrm{Sub}\, X \to \mathrm{Sub}\, Y$ to the pullback functor f^*. These functors play a central role in interpretations of logic into these categories, as we will see in detail shortly.

Moreover, for such categories $\mathcal{C}$ with extra structure it may be the case that for each $X \in \mathcal{C}$, $\mathrm{Sub}\, X$ is in fact a *Heyting algebra*. That is, not only does $\mathrm{Sub}\, X$ possess the structure of a lattice, but also for every $A, B \in \mathrm{Sub}\, X$, there is $(A \Rightarrow B) \in \mathrm{Sub}\, X$ ("implication") such that for every $C \in \mathrm{Sub}\, X$, we have that $C \leq (A \Rightarrow B)$ iff $C \wedge A \leq B$.

It is not true in general that a category $\mathcal{C}$ admits such extra structure. This is the case, however, if $\mathcal{C}$ is a *topos*, which is a category satisfying some strong conditions; **Set**, for example, is a topos. We will not delve into topos theory in this thesis, for we will construct what we need by hand.

7.3 The categorical interpretation

Let us consider the syntactic framework as described in Section 7.1.

Definition 7.7. Let S be a signature as in Section 7.1.

Let $\mathcal{C}$ be a category with finite limits and colimits. Furthermore for each $X \in \mathcal{C}$ let

$\mathrm{Sub}\, X$ possess the structure of a Heyting algebra (whose lattice structure is given as in Proposition 7.4), and for each $X, Y \in \mathcal{C}$ and $f : X \to Y$ let there be a left (resp. right) adjoint $\exists_f$ (resp. $\forall_f$) : $\mathrm{Sub}\, Y \to \mathrm{Sub}\, X$.

An *interpretation of S in $\mathcal{C}$* is given by the following:

(a) For each sort symbol $s \in \mathcal{S}$, an object $[\![s]\!] \in \mathcal{C}$, such that $[\![*]\!]$ is the terminal object of $\mathcal{C}$.

(b) For each function symbol $f : \prod_i s_i \to s$, a morphism $[\![f]\!] : \prod_i [\![s_i]\!] \to [\![s]\!]$.

(c) For each predicate symbol $R \subset \prod_i s_i$, a subobject $[\![R]\!] \hookrightarrow \prod_i [\![s_i]\!]$.

(In the above we have implicitly made a choice of products, and monomorphisms representing subobjects.)

The above data then determine the interpretation of all S-terms and S-formulas, as follows. Any time we have a tuple $\vec{x} = (x_1, \ldots, x_n)$ of distinct variables of types $s_1, \ldots, s_n$ (respectively), we set $[\![\vec{x}]\!] = \prod_{1 \le i \le n} [\![s_i]\!]$. In particular, if a variable x is of type s then $[\![x]\!]$ is $[\![s]\!]$, and if $\vec{x}$ is empty then $[\![\vec{x}]\!] = [\![*]\!]$.

If t is a term of type s with free variables among $\vec{x}$, then $[\![t]\!]_{\vec{x}}$ is defined as a morphism $[\![t]\!]_{\vec{x}} : [\![\vec{x}]\!] \to [\![s]\!]$ given by the following inductive description:

If $t = x_i$ then $[\![t]\!]_{\vec{x}} : [\![\vec{x}]\!] \to [\![x_i]\!]$ is just the projection map.

If $t = f(t_1, \ldots, t_n)$ with each t_i of type s_i, with each $[\![t_i]\!]_{\vec{x}}$ already defined, then $[\![t]\!]_{\vec{x}} : [\![\vec{x}]\!] \to [\![s]\!]$ is given by the composition $[\![f]\!] \circ ([\![t_1]\!]_{\vec{x}}, \ldots, [\![t_n]\!]_{\vec{x}})$.

Now if ϕ is a formula with free variables among $\vec{x}$, then we interpret it as a subobject $[\![\phi]\!]_{\vec{x}} \hookrightarrow [\![\vec{x}]\!]$, given by the following:

If ϕ is the atomic formula $t_1 = t_2$, with both t_1 and t_2 are terms of type s, then $[\![\phi]\!]_{\vec{x}}$ is given by the equalizer $[\![\phi]\!]_{\vec{x}} \longrightarrow [\![\vec{x}]\!] \overset{[\![t_1]\!]_{\vec{x}}}{\underset{[\![t_2]\!]_{\vec{x}}}{\rightrightarrows}} [\![s]\!]$.

If ϕ is an atomic formula $R(t_1, \ldots, t_n)$ where t_i has sort s_i, then $[\![\phi]\!]_{\vec{x}}$ is given by the pullback

$$\begin{tikzcd}
[\![\phi]\!]_{\vec{x}} \arrow[r] \arrow[d] & [\![R]\!] \arrow[d] \\
[\![\vec{x}]\!] \arrow[r, "([\![t_1]\!], \ldots, [\![t_n]\!])"] & \prod_i [\![s_i]\!]
\end{tikzcd}$$

Given interpretations $[\![\phi]\!]_{\vec{x}}$ and $[\![\psi]\!]_{\vec{x}}$ of ϕ and ψ, we set $[\![\phi \wedge \psi]\!]_{\vec{x}} = [\![\phi]\!]_{\vec{x}} \wedge [\![\psi]\!]_{\vec{x}}$, $[\![\phi \vee \psi]\!]_{\vec{x}} = [\![\phi]\!]_{\vec{x}} \vee [\![\psi]\!]_{\vec{x}}$, and $[\![\phi \Rightarrow \psi]\!]_{\vec{x}} = [\![\phi]\!]_{\vec{x}} \Rightarrow [\![\psi]\!]_{\vec{x}}$.

Given an interpretation $[\![\phi]\!]_{\vec{x},y}$ of ϕ, we set $[\![\exists y\, \phi]\!]_{\vec{x}} = \exists_\pi\left([\![\phi]\!]_{\vec{x},y}\right)$ and $[\![\forall y\, \phi]\!]_{\vec{x}} = \forall_\pi\left([\![\phi]\!]_{\vec{x},y}\right)$, where $\pi : (\prod_i [\![\vec{x}]\!]) \times [\![y]\!] \to \prod_i [\![\vec{x}]\!]$ is the obvious projection.

This completes the interpretation of all S-terms and S-formulas into the category $\mathcal{C}$. From the above, we see that if ϕ is a sentence, then $[\![\phi]\!]_\emptyset \hookrightarrow [\![*]\!]$ is a subobject of the terminal object. We say that ϕ is *true* in $\mathcal{C}$ if this arrow $[\![\phi]\!]_\emptyset \hookrightarrow [\![*]\!]$ is in fact an isomorphism, i.e. $[\![\phi]\!]_\emptyset$ is itself the terminal object.

Then if Σ is a collection of S-sentences, we say that an interpretation of S into $\mathcal{C}$ is a *model* of Σ if the interpretation makes each $\phi \in \Sigma$ true in $\mathcal{C}$.

7.4 Some extras about the category Set

One observation about the category **Set** is that, given $X \in \mathbf{Set}$, each $A \in \mathcal{P}(X)$ corresponds uniquely to a map $f : X \to \{0, 1\}$ where $A = f^{-1}(\{0\})$. We can generalize this to general categories:

Definition 7.8. For $\mathcal{C}$ a category with finite limits (and therefore a terminal object $* \in \mathcal{C}$), we say that $* \hookrightarrow \Omega$ (for some $\Omega \in \mathcal{C}$) is a *subobject classifier* when for each $\iota : A \hookrightarrow X$ there is a unique morphism $f : X \to \Omega$ such that the following is a pullback diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, hook] \\
X \arrow[r, "f"] & \Omega
\end{tikzcd}$$

So for $\mathcal{C} = \mathbf{Set}$ we have $(* \hookrightarrow \Omega) = (\{0\} \hookrightarrow \{0,1\})$. This correspondence between subobjects and maps to the subobject classifier is what allows us to consider formulas ϕ as either subsets $[\![\phi]\!]_{\vec{x}} \subset [\![\vec{x}]\!]$ or as functions $\phi : [\![\vec{x}]\!] \to \{0,1\}$. Furthermore we can consider an ordering on Ω, namely $0 < 1$, so that given functions $f_1, f_2 : X \to \Omega$ we have $\max(f_1, f_2) : X \to \Omega$ and $\min(f_1, f_2) : X \to \Omega$.

Proposition 7.9. *In the category* $\mathbf{Set}$*, let ϕ_A and ϕ_B be interpreted as $[\![\phi_A]\!]_{\vec{x}} \subset [\![\vec{x}]\!]$ and $[\![\phi_B]\!]_{\vec{x}} \subset [\![\vec{x}]\!]$, respectively.*

Denote $[\![\vec{x}]\!]$ by X, and let $A \hookrightarrow X$ and $B \hookrightarrow X$ represent $[\![\phi_A]\!] \subset X$ and $[\![\phi_B]\!] \subset X$, respectively.

Let $A \hookrightarrow X$ and $B \hookrightarrow X$ correspond to $f_A : X \to \Omega$ and $f_B : X \to \Omega$, respectively.

(a) $[\![\phi_A \wedge \phi_B]\!] \subset X$ is represented by $A \cap B \hookrightarrow X$, which corresponds to

$\max(f_A, f_B) : X \to \Omega.$

(b) $[\![\phi_A \vee \phi_B]\!] \subset X$ is represented by $A \cup B \hookrightarrow X$, which corresponds to

$\min(f_A, f_B) : X \to \Omega.$

With the above we have described how acting on formulas by connectives affects their interpretations into **Set**, either as subsets or as maps to $\{0, 1\}$. We wish to do the same for quantification.

Proposition 7.10. *Let ϕ be a formula with free variables among $\vec{x}, y$, and denote $X = [\![\vec{x}]\!]$ and $A = [\![\phi]\!]_{\vec{x},y} \subset X \times [\![y]\!]$. We have the projection $\pi : X \times [\![y]\!] \to X$. Furthermore, let ϕ correspond to $f_A : (X \times [\![y]\!]) \to \Omega$.*

(a) $[\![\forall y\, \phi]\!]_{\vec{x}}$ is given by the set $\{x \mid (\pi)^{-1}(\{x\}) \subset A\} \subset X$, which corresponds to the function $(\forall_\pi f) : X \to \Omega$ given by $a \mapsto \sup_{b \in [\![y]\!]} f_A(a, b)$.

(b) $[\![\exists y\, \phi]\!]_{\vec{x}}$ is given by the set $\pi(A) \subset X$, which corresponds to the function $(\exists_\pi f) : X \to \Omega$ given by $a \mapsto \inf_{b \in [\![y]\!]} f_A(a, b)$.

Chapter 8

Some features of enriched categories

Ultimately our goal is to give a categorical interpretation of continuous first order logic in an analogous manner to that described above for classical first order logic. Our approach to this is inspired by Lawvere, who in [27] investigates various logically relevant properties of enriched categories and how the enriching category affects the nature of those properties. A reference for enriched category theory is e.g. [14]. However, for our purposes we will need to suitably tailor the theory of enriched categories to fit our needs. As the currently existing framework of enriched category theory is vast, it falls outside the scope of this thesis to track every consequence of these modifications throughout the entirety of the existing framework; we will therefore flesh out in complete detail only those parts of enriched category theory that we require for our current objectives, and how those parts of the theory are affected under our modifications.

For the most part we will sweep under the rug issues of size, as we will assume we are working in a Grothendieck universe of sufficient size to accomodate the constructions we need. More precisely, we potentially need up to two inaccessible cardinals $\kappa < \kappa'$, when we work with such constructions as "the category of the κ'-small categories of κ-small $\mathcal{V}$-enriched categories".

Section 8.1 recalls some standard notions in enriched category theory, while also introducing some natural but non-standard definitions that we will utilize: we take care to distinguish in each case whether it is a standard concept or our own addition to the theory. Section 8.2 uses the material introduced in Section 8.1 to note some relevant properties of enriched categories that are analogous to those of ordinary categories. Section 8.3 illustrates how, as a special case, the category of sets may be considered as a category of enriched categories, to suggest how the material of this chapter will (after suitable modification) be utilized in the following chapter in pursuing the goal of interpreting logic into enriched categories.

8.1 Preliminaries

We briefly recall some standard notions concerning enriched categories that we will use:

Let $(\mathcal{V}, \otimes, I)$ be a symmetric closed monoidal category that is complete (has all limits) and cocomplete (has all colimits); for convenience we call such a category a *cosmos*. We additionally assume that I is the terminal object for $\mathcal{V}$ (which is usually the case).

A $\mathcal{V}$-category X is given by specifiying, for every three objects $a, b, c \in X$, a morphism $\mu^X_{a,b,c} : X(b, c) \otimes X(a, b) \to X(a, c)$ in $\mathcal{V}$, and also for each $a \in X$ a morphism

$\eta_a^X : I \to X(a,a)$ in $\mathcal{V}$ such that the following diagram commutes:

$$\begin{tikzcd}
X(a,b) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] \arrow[ddrr, "1_{X(a,b)}"] & X(a,b) \otimes I \arrow[r, "X(a,b) \otimes \eta_a"] & X(a,b) \otimes X(a,a) \arrow[dd, "\mu_{a,a,b}"] \\
I \otimes X(a,b) \arrow[d, "\eta_b \otimes X(a,b)"'] & & \\
X(b,b) \otimes X(a,b) \arrow[rr, "\mu_{a,b,b}"] & & X(a,b)
\end{tikzcd}$$

That $\mathcal{V}$ is closed means that we have a functor $\mathrm{Hom}_{\mathcal{V}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ such that there is a bijection between the set $\mathcal{V}_0(b \otimes a, c)$ of $\mathcal{V}$-morphisms $b \otimes a \to c$ and the set $\mathcal{V}_0(a, \mathrm{Hom}_{\mathcal{V}}(b,c))$ of $\mathcal{V}$-morphisms $a \to \mathrm{Hom}_{\mathcal{V}}(b,c)$ which satisfies the evident naturality conditions in a, b, c. This makes $\mathcal{V}$ itself into a $\mathcal{V}$-category, by setting $\mathcal{V}(a,b) = \mathrm{Hom}_{\mathcal{V}}(a,b)$.

Given a $\mathcal{V}$-category X we can take X^{op} to be the $\mathcal{V}$-category with the same objects and $X^{\mathrm{op}}(a,b) = X(b,a)$. We call X *symmetric* when $X = X^{\mathrm{op}}$. Given an arbitrary $\mathcal{V}$-category X we can always *symmetrize* it to get X_{sym}, the $\mathcal{V}$-category with the same objects and $X_{\mathrm{sym}}(a,b) = X(b,a) \otimes X(a,b)$.

Remark 8.1. If $\mathcal{V}$ is a linear order then the definition of $\mathcal{V}$ as a $\mathcal{V}$-category necessarily implies that $\mathcal{V}_{\mathrm{sym}}(a,b) = \mathcal{V}(a,b) \otimes \mathcal{V}(b,a) = \mathcal{V}(a,b) \times \mathcal{V}(b,a)$ since at least one of $\mathcal{V}(a,b)$ or $\mathcal{V}(b,a)$ must be equal to I.

We now give a definition that is (to the best of the author's knowledge) is not standard, but is a natural generalization of its counterpart in ordinary category theory:

Definition 8.2. Given a $\mathcal{V}$-category X, we say that two objects $a, b \in X$ are *$\mathcal{V}$-isomorphic* when there are $\mathcal{V}$-morphisms $i_{a,b} : I \to X(a,b)$ and $i_{b,a} : I \to X(b,a)$ such that the following diagram commutes:

$$\begin{tikzcd}
X(b,a) \otimes X(a,b) \arrow[rr, "\mu_{a,b,a}"] \arrow[dd, "\simeq"'] & & X(a,a) \\
& I \arrow[ul, "i_{b,a} \oplus i_{a,b}"'] \arrow[ur, "\eta_a"'] \arrow[dl, "i_{a,b} \oplus i_{b,a}"] \arrow[dr, "\eta_b"] & \\
X(a,b) \otimes X(b,a) \arrow[rr, "\mu_{b,a,b}"'] & & X(b,b)
\end{tikzcd}$$

where the leftmost isomorphism is the symmetry of the monoidal structure in $\mathcal{V}$.

It follows easily that if $a, b \in X$ are $\mathcal{V}$-isomorphic then $X(c,a) \simeq X(c,b)$ and $X(a,c) \simeq X(b,c)$ in $\mathcal{V}$ for all $c \in X$.

The following is a standard notion in enriched category theory:

If X and Y are $\mathcal{V}$-categories, then by a $\mathcal{V}$-functor $F : X \to Y$ we mean an assignment of a Y-object Fa to each X-object a along with a $\mathcal{V}$-morphism $F_{a,b} : X(a,b) \to Y(Fa, Fb)$ for each pair of X-objects a, b such that the following diagrams commute:

$$\begin{tikzcd}
X(a,b) \otimes X(b,c) \arrow[r, "\mu^X_{a,b,c}"] \arrow[d, "F_{a,b} \otimes F_{b,c}"'] & X(a,c) \arrow[d, "F_{a,c}"] \\
Y(Fa,Fb) \otimes Y(Fb,Fc) \arrow[r, "\mu^Y_{Fa,Fb,Fc}"] & Y(Fa,Fc)
\end{tikzcd}$$

$$\begin{tikzcd}
& X(a,a) \arrow[d, "F_{a,a}"] \\
I \arrow[ur, "\eta^X_a"] \arrow[dr, "\eta^Y_{Fa}"'] & \\
& Y(Fa,Fa)
\end{tikzcd}$$

A short diagram chase shows that F must take $\mathcal{V}$-isomorphic objects in X to $\mathcal{V}$-isomorphic objects in Y.

A $\mathcal{V}$-functor F is called *$\mathcal{V}$-f.f.* (for "$\mathcal{V}$-full and faithful") when $F_{a,b} : X(a,b) \to Y(Fa, Fb)$ is an isomorphism in $\mathcal{V}$ for each $a, b \in X$. If F is also injective on objects then we call it an *embedding of $\mathcal{V}$-categories*, and if F is furthermore a bijection on objects then we call it an *isomorphism of $\mathcal{V}$-categories*, or a ($\mathcal{V}$-**Cat**)*-isomorphism.*

We call a $\mathcal{V}$-functor $F : X \to Y$ *saturated* when for every $x \in X$ and every $y \in Y$ such that y is $\mathcal{V}$-isomorphic to Fx in Y, there is some $x' \in X$ such that $Fx' = y$.

Keeping in mind that our approach is to combine the Makkai-Reyes categorical interpretation of logic [33] with Lawvere's method of enriching the interpreting category in a suitable poset of truth values [27], we now make the significant assumption that our enriching category $\mathcal{V}$ is in fact also a poset category. This will not be a hindrance for us since we would have ended up specializing to $\mathcal{V}$ a poset category anyway, and this assumption makes some parts of the machinery we need considerably simpler to develop and use.

As a first consequence of this assumption, note that for X a $\mathcal{V}$-category we have that $a, b \in X$ are $\mathcal{V}$-isomorphic iff $X(a,b) = I$.

In what follows, recall that we are assuming up to two inaccessible cardinals $\kappa < \kappa'$; when we speak of such things as categories of small categories, we mean a κ'-small category of κ-small categories, justifying the use of set-theoretic language at our convenience.

Consider $\mathcal{V}$-**Cat**, the category of small $\mathcal{V}$-enriched categories, where the objects are $\mathcal{V}$-categories and morphisms $\mathcal{V}$-functors between them. $\mathcal{V}$-**Cat** is a priori just an ordinary category: for $X, Y \in \mathcal{V}$-**Cat** we have $\mathcal{V}$-$\mathbf{Cat}_0(X,Y)$ given by the set of $\mathcal{V}$-functors from X to Y. However there is a standard end formula that endows $\mathcal{V}$-$\mathbf{Cat}_0(X,Y)$ with the structure of a $\mathcal{V}$-category; specifically, for $F, G \in \mathcal{V}\text{-}\mathbf{Cat}_0(X, Y)$, we can take $\mathcal{V}\text{-}\mathbf{Cat}(X, Y)(F, G)$ to be the end $\int_a Y(Fa, Ga) \in \mathcal{V}$.

Explicitly, $\int_a Y(Fa, Ga)$ is obtained as the equalizer

$$\prod_a Y(Fa, Ga) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{b,c} \mathrm{Hom}_{\mathcal{V}}(X(b, c), Y(Fb, Gc))$$

where α is induced from the morphisms

$$Y(Fc, Gc) \otimes X(b, c) \to Y(Fc, Gc) \otimes X(Fb, Fc) \to Y(Fb, Gc)$$

and β from the morphisms

$$X(b, c) \otimes Y(Fb, Gb) \to Y(Gb, Gc) \otimes Y(Fb, Gb) \to Y(Fb, Gc)$$

via the tensor-hom adjunction in $\mathcal{V}$.

Since we are assuming $\mathcal{V}$ is a poset category, the equalizer above simplifies to a product, and $\mathcal{V}\text{-}\mathbf{Cat}(X, Y)(F, G) = \prod_a Y(Fa, Ga)$.

The tensor product on $\mathcal{V}$-**Cat** is inherited from that of $\mathcal{V}$: for $X, Y \in \mathcal{V}\text{-}\mathbf{Cat}$ one takes $X \otimes Y$ to be the $\mathcal{V}$-category with objects (a, b) where $a \in X$ and $b \in Y$, and

$$(X \otimes Y)((a_1, b_1), (a_2, b_2)) = X(a_1, a_2) \otimes Y(b_1, b_2).$$

The monoidal unit of $\mathcal{V}$-**Cat** is the category $\mathbb{I}$ with one object $*$ where $\mathbb{I}(*, *) = I$.

For $X, Y, Z \in \mathcal{V}$-**Cat**, we clearly have the composition map

$$\mathcal{V}\text{-}\mathbf{Cat}_0(Y,Z) \times \mathcal{V}\text{-}\mathbf{Cat}_0(X,Y) \to \mathcal{V}\text{-}\mathbf{Cat}_0(X,Z)$$

at the set level. It turns out that this is the "underlying set" of a $\mathcal{V}$-functor $\mathcal{V}$-functor $\mathcal{V}\text{-}\mathbf{Cat}(Y,Z) \otimes \mathcal{V}\text{-}\mathbf{Cat}(X,Y) \to \mathcal{V}\text{-}\mathbf{Cat}(X,Z)$:

Proposition 8.3. *Let $X, Y, Z \in \mathcal{V}$-**Cat**.*

The set map $\mathcal{V}\text{-}\mathbf{Cat}_0(Y,Z) \times \mathcal{V}\text{-}\mathbf{Cat}_0(X,Y) \to \mathcal{V}\text{-}\mathbf{Cat}_0(X,Z)$ given by composition of $\mathcal{V}$-functors is the underlying object function of a $\mathcal{V}$-functor

$$\mathcal{V}\text{-}\mathbf{Cat}(Y,Z) \otimes \mathcal{V}\text{-}\mathbf{Cat}(X,Y) \to \mathcal{V}\text{-}\mathbf{Cat}(X,Z).$$

Remark 8.4. This proposition, as well as its proof, is surely known; however the author has not been able to find it in the literature, so we record it here for completeness.

Proof. Let $F_1, F_2 \in \mathcal{V}\text{-}\mathbf{Cat}(X,Y)$ and $G_1, G_2 \in \mathcal{V}\text{-}\mathbf{Cat}(Y,Z)$. For simplicity denote $\mathcal{V}\text{-}\mathbf{Cat}(X,Y)$ by Ω_1, $\mathcal{V}\text{-}\mathbf{Cat}(Y,Z)$ by Ω_2, and $\mathcal{V}\text{-}\mathbf{Cat}(X,Z)$ by Ω. We wish to give a morphism

$$\Omega_2 \otimes \Omega_1((F_1,G_1),(F_2,G_2)) = \Omega_2(G_1,G_2) \otimes \Omega_1(F_1,F_2) \to \Omega(G_1F_1, G_2F_2)$$

in $\mathcal{V}$. This means that we need to give a morphism

$$\left(\prod_y Z(G_1y, G_2y)\right) \otimes \left(\prod_x Y(F_1x, F_2x)\right) \to \prod_x Z(G_1F_1x, G_2F_2x)$$

in $\mathcal{V}$. Let Y' denote the set of objects of Y that are *not* of the form F_2x for any $x \in X$, and let X' denote a set of objects of X on which F_2 is a bijection onto its whole image. Then we have that $\prod_y Z(G_1y, G_2y) = \left(\prod_{y \in Y'} Z(G_1y, G_2y)\right) \times \left(\prod_{x \in X'} Z(G_1F_2x, G_2F_2x)\right)$. However, since $\mathcal{V}$ is a poset we have that for any $a \in \mathcal{V}$, $a \simeq \prod_{i \in J} a$ for any nonempty index set J, so that $\prod_{x \in X'} Z(G_1F_2x, G_2F_2x) \simeq \prod_x Z(G_1F_2x, G_2F_2x)$.

Then we have the following morphisms in $\mathcal{V}$:

$$\left(\prod_{y \in Y'} Z(G_1y, G_2y)\right) \times \left(\prod_x Z(G_1F_2x, G_2F_2x)\right) \xrightarrow{\pi_2} \prod_x Z(G_1F_2x, G_2F_2x)$$

$$\prod_x Y(F_1x, F_2x) \xrightarrow{\prod_x (G_1)_{F_1x, F_2x}} \prod_x Z(G_1F_1x, G_1F_2x)$$

Now for each $x \in X$ we have a map

$$\prod_x Z(G_1F_2x, G_2F_2x) \otimes \prod_x Z(G_1F_1x, G_1F_2x) \xrightarrow{\pi_x \otimes \pi_x} Z(G_1F_2x, G_2F_2x) \otimes Z(G_1F_1x, G_1F_2x)$$
$$\xrightarrow{\mu^Z} Z(G_1F_1x, G_2F_2x)$$

which gives us a morphism

$$\prod_x Z(G_1F_2x, G_2F_2x) \otimes \prod_x Z(G_1F_1x, G_1F_2x) \to \prod_x Z(G_1F_1x, G_2F_2x) = \Omega(G_1F_1, G_2F_2)$$

Putting all of the above together gives us the required $\mathcal{V}$-morphism

$$\mathcal{V}\text{-}\mathbf{Cat}(Y, Z) \otimes \mathcal{V}\text{-}\mathbf{Cat}(X, Y)((F_1, G_1), (F_2, G_2)) \to \mathcal{V}\text{-}\mathbf{Cat}(X, Z)(G_1F_1, G_2F_2)$$

for each $F_1, F_2 \in \mathcal{V}\text{-}\mathbf{Cat}(X, Y)$ and $G_1, G_2 \in \mathcal{V}\text{-}\mathbf{Cat}(Y, Z)$. $\mathcal{V}$-functoriality is immediate since $\mathcal{V}$ is a poset, so that we have a $\mathcal{V}$-functor

$$\mathcal{V}\text{-}\mathbf{Cat}(Y, Z) \otimes \mathcal{V}\text{-}\mathbf{Cat}(X, Y) \to \mathcal{V}\text{-}\mathbf{Cat}(X, Z)$$

which agrees at the object level with the set function

$$\mathcal{V}\text{-}\mathbf{Cat}_0(Y, Z) \times \mathcal{V}\text{-}\mathbf{Cat}_0(X, Y) \to \mathcal{V}\text{-}\mathbf{Cat}_0(X, Z)$$

$\square$

In particular, $\mathcal{V}$-**Cat** can be considered as enriched over itself.

In light of this, for any $\mathcal{V}$-**Cat**-enriched category $\mathcal{A}$ and $X, Y, Z \in \mathcal{A}$, if we have $F \in \mathcal{A}(X, Y)$ and $G \in \mathcal{A}(Y, Z)$ then we will write $GF \in \mathcal{A}(X, Z)$ for the image under composition map of the object $G \otimes F \in \mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y)$.

An important consequence of the composition map being a $\mathcal{V}$-functor is that precomposing or postcomposing (by any $\mathcal{V}$-functor) preserves the relation of two $\mathcal{V}$-functors being $\mathcal{V}$-isomorphic.

We can now define the following, which - like the notion of being $\mathcal{V}$-isomorphic - is not entirely standard to the best of the author's knowledge, but is a natural enriched version of its counterpart in ordinary category theory:

Definition 8.5. Given $X, Y \in \mathcal{V}$-**Cat**, we say that X, Y are $\mathcal{V}$-*equivalent* when there are $\mathcal{V}$-functors $F : X \to Y$ and $G : Y \to X$ such that GF is $\mathcal{V}$-isomorphic to 1_X in $\mathcal{V}$-**Cat**(X, X) and FG is $\mathcal{V}$-isomorphic to 1_Y in $\mathcal{V}$-**Cat**(Y, Y). We may denote this situation as $X \underset{G}{\overset{F}{\rightleftarrows}} Y$.

In general, we will call $\mathcal{V}$-functors $F, G : X \to Y$ $\mathcal{V}$-isomorphic when they are $\mathcal{V}$-isomorphic as objects of the $\mathcal{V}$-category $\mathcal{V}\text{-}\mathbf{Cat}(X, Y)$. Moreover, if we have $X \underset{F'}{\overset{F}{\rightleftarrows}} X'$ and $Y \underset{H'}{\overset{H}{\rightleftarrows}} Y'$ then we have that for $G : X \to Y$, G is $\mathcal{V}$-isomorphic to $H'HGF'F$.

This motivates the following convention, that when we say two $\mathcal{V}$-functors $G : X \to Y$ and $G' : X' \to Y'$ are $\mathcal{V}$-*equivalent*, we mean that there is a $\mathcal{V}$-equivalence $X \underset{F'}{\overset{F}{\rightleftarrows}} X'$ and a $\mathcal{V}$-equivalence $Y \underset{H'}{\overset{H}{\rightleftarrows}} Y'$ such that G' is $\mathcal{V}$-isomorphic to HGF' ($\Leftrightarrow G$ is $\mathcal{V}$-isomorphic to $H'G'F \Leftrightarrow HG$ is $\mathcal{V}$-isomorphic to $G'F \Leftrightarrow GF'$ is $\mathcal{V}$-isomorphic to $H'G'$). If we take $F = F' = 1_X$ then we say that this $\mathcal{V}$-equivalence of functors *fixes* X, while if $G = G' = 1_Y$ then we say that the $\mathcal{V}$-equivalence *fixes* Y.

Although $\mathcal{V}\text{-}\mathbf{Cat}$ is the main example of a $(\mathcal{V}\text{-}\mathbf{Cat})$-enriched category, the above definitions generalize easily to any $(\mathcal{V}\text{-}\mathbf{Cat})$-enriched category $\mathcal{A}$: instead of $\mathcal{V}$-functors $F : X \to Y$ we have objects $F \in \mathcal{A}(X, Y)$ (with the identity $\mathcal{V}$-functor 1_X being replaced by the object of $\mathcal{A}(X, X)$ given by the unit $\mathbb{I} \to \mathcal{A}(X, X)$), and instead of the composition $\mathcal{V}$-functor $\mathcal{V}\text{-}\mathbf{Cat}(Y, Z) \otimes \mathcal{V}\text{-}\mathbf{Cat}(X, Y) \to \mathcal{V}\text{-}\mathbf{Cat}(X, Z)$ we consider the $\mathcal{V}$-functor $\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$.

8.2 Subobjects of $\mathcal{V}$-categories

We now define what it means to be a subobject:

Definition 8.6. Let X be a $\mathcal{V}$-enriched category. By a *subobject* of X we mean the equivalence class of a saturated embedding $\iota : A \to X$, where $\iota : A \to X$ is considered equivalent to $\iota' : A' \to X$ when there is an isomorphism of $\mathcal{V}$-categories given by $F : A \to A'$ such that ι is $\mathcal{V}$-isomorphic to $\iota' F$.

We will often refer to a subobject of X by (the domain of) one of its representatives, i.e.

"$\iota : A \to X$ is a subobject of X" or "A is a subobject of X" when no confusion will result.

We will also need to talk of "subcategories" in $\mathcal{V}$-**Cat**, so we introduce the following terminology:

Definition 8.7. Let $\mathcal{A}$ and $\mathcal{B}$ be κ'-small ($\mathcal{V}$-**Cat**)-enriched categories, then we say that $\mathcal{A}$ is a *subcategory* of $\mathcal{B}$ if there is a ($\mathcal{V}$-**Cat**)-functor $i : \mathcal{A} \to \mathcal{B}$ that is injective on objects, such that for each $X, Y \in \mathcal{A}$ the $\mathcal{V}$-functor $i_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{B}(i(X), i(Y))$ is an embedding of $\mathcal{V}$-categories. If each such $i_{X,Y}$ is in fact an isomorphism of $\mathcal{V}$-categories then we call $\mathcal{A}$ a *full subcategory* of $\mathcal{B}$.

An important example when $\mathcal{B} = \mathcal{V}\text{-}\mathbf{Cat}$ is $\mathcal{A} = (\mathcal{V}\text{-}\mathbf{Cat})_{\mathrm{sym}}$, the full subcategory of symmetric $\mathcal{V}$-categories. Tracing the argument of Theorem 8.3 shows us that $(\mathcal{V}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ is not just enriched over $\mathcal{V}$-**Cat** but that this enrichment restricts to the subcategory $(\mathcal{V}\text{-}\mathbf{Cat})_{\mathrm{sym}}$. This is a regularly occurring theme: *all of the results below that reference* $\mathcal{V}$-**Cat** *should be understood as also applying to the full subcategory* $(\mathcal{V}\text{-}\mathbf{Cat})_{sym}$ *(with the same proofs), unless stated otherwise.*

8.2.1 Limits and factorization in $\mathcal{V}$-Cat

We now need an appropriate notion of limits in $\mathcal{V}$ that generalizes the notion of limits in an ordinary category. There is already a sophisticated tool that does this, called *weighted (co)limits*; however, to apply this theory to limits in $\mathcal{V}$-**Cat** would require us to talk of hom objects in ($\mathcal{V}$-**Cat**)-**Cat**, the construction of which only considers the ordinary categorical structure of $\mathcal{V}$-**Cat** and moreover complicates matters more than is necessary. Therefore, we take a more elementary approach that nevertheless specializes appropriately to the familiar notion of limits in the case of ordinary categories.

Definition 8.8. Let $\mathcal{A}, \mathcal{B}$ be κ'-small ($\mathcal{V}$-**Cat**)-enriched categories, and let $D : \mathcal{A} \to \mathcal{B}$ be a ($\mathcal{V}$-**Cat**)-functor, which we call a diagram.

By a *cone* λ *over* D we mean an object $X \in \mathcal{B}$ along with, for each $A \in \mathcal{A}$, an object $\lambda_A \in \mathcal{B}(X, DA)$ such that for every $F \in \mathcal{A}(A_1, A_2)$, we have that $(D_{A_1,A_2}F)\lambda_{A_1}$ is $\mathcal{V}$-isomorphic to λ_{A_2} in $\mathcal{B}(X, DA_2)$. We may also call λ a *cone from* X *to* D.

By a *limit of* D we mean an object $\lim D \in \mathcal{B}$ along with a cone λ from $\lim D$ to D such that for any $X \in \mathcal{B}$ and a cone μ from X to D, there is some $L \in \mathcal{B}(X, \lim D)$, unique up to $\mathcal{V}$-isomorphism, such that for each $A \in \mathcal{A}$, we have that μ_A is $\mathcal{V}$-isomorphic to $\lambda_A L$ in $\mathcal{B}(X, DA)$.

If there is a choice of $\lim D$ along with a choice of limit cone λ satisfying the commutativity conditions strictly and not only up to $\mathcal{V}$-isomorphism; and further if for each $X \in \mathcal{B}$ with a cone μ from X to D there exists an $L \in \mathcal{B}(X, \lim D)$ for each $A \in \mathcal{A}$ we have that $\mu_A = \lambda_A L$ strictly, then we say that this choice of $\lim D$ has the *strict universal property.*

Cocones and colimits are defined dually.

We note that, with notation as in the above definition, $\lim D$ must be unique up to $\mathcal{V}$-equivalence (if it exists). Also, if the image of some $D' : \mathcal{A} \to \mathcal{B}$ is $\mathcal{V}$-equivalent to the image of D in the sense that for each $A_1, A_2 \in \mathcal{A}$ and each $F : A_1 \to A_2$ we have that $DF : DA_1 \to DA_2$ is $\mathcal{V}$-equivalent to $D'F : D'A_1 \to D'A_2$, then $\lim D$ is $\mathcal{V}$-equivalent to $\lim D'$.

However, we should be careful to note that $\mathcal{V}$-equivalence does not preserve the *strict* universal property, although isomorphisms of $\mathcal{V}$-categories do.

Furthermore, if $\mathcal{A}$ is the empty ($\mathcal{V}$-**Cat**)-category and $\mathcal{B} = \mathcal{V}$-**Cat** then we can take $\lim D = \mathbb{I}$; thus the monoidal unit of $\mathcal{V}$-**Cat** is also its *terminal object.*

If $\mathcal{A}$ is taken to be the ($\mathcal{V}$-**Cat**)-enriched category with three objects A_1, A_2, A_3 with $\mathcal{A}(A_1, A_1) = \mathcal{A}(A_2, A_2) = \mathcal{A}(A_3, A_3) = \mathcal{A}(A_1, A_3) = \mathcal{A}(A_2, A_3) = \mathbb{I}$ and empty hom-objects otherwise, then $\lim D$ is also called a *pullback*. ($\mathcal{A}$ is illustrated by the diagram below, where we have omitted the "identity $\mathcal{V}$-functors".)

$$\begin{tikzcd} & A_1 \arrow[d] \\ A_2 \arrow[r] & A_3 \end{tikzcd}$$

If X is a $\mathcal{V}$-category, there is a $\mathcal{V}$-category X_0 with a $\mathcal{V}$-equivalence $X \underset{i}{\overset{\pi}{\rightrightarrows}} X_0$ such that every $x \in X_0$ is the only member of its $\mathcal{V}$-isomorphism class; we call this the *reduction* of X, and π the *reduction map*. Explicitly, X_0 has as objects the $\mathcal{V}$-isomorphism classes of objects in X, and π is the $\mathcal{V}$-f.f. $\mathcal{V}$-functor that sends each object in X to its $\mathcal{V}$-isomorphism class.

Proposition 8.9. $\mathcal{V}$-**Cat** *has pullbacks.*

Proof. Let $X, Y, Z \in \mathcal{V}$-**Cat** and let $F : X \to Z$ and $G : Y \to Z$ be $\mathcal{V}$-functors fitting into the below diagram:

$$\begin{tikzcd} & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

We construct a $\mathcal{V}$-category A and $\mathcal{V}$-functors $U : A \to X$ and $V : A \to Y$ that satisfy the conditions for being a limit of the above diagram.

First note that if $Z = \mathbb{I}$ then we can take A to be the product $X \times Y$, which has objects pairs of the form (x, y) for $x \in X$ and $y \in Y$, with $X \times Y((x, y), (x', y')) = X(x, x') \times Y(y, y')$ (where the product is taken in $\mathcal{V}$). Then U and V are just the obvious projections p_X and p_Y.

Let Z_0 denote the reduction of Z and $\pi : Z \to Z_0$ the reduction map.

The set of objects of A is given by the set $\coprod_{z \in Z_0} \big((\pi F)^{-1}(z) \times (\pi G)^{-1}(z)\big)$. This is clearly a subset of the set of objects of $X \times Y$, so that every $a \in A$ is uniquely of the form (x, y) for $x \in X$ and $y \in Y$. Then for $a = (x, y), a' = (x', y')$ in A, we set $A(a, a') = X(x, x') \times Y(y, y')$ (where the product is taken in $\mathcal{V}$). U and V are simply the restrictions of the projections p_X and p_Y to A.

$\square$

Proposition 8.10. *Let $X, Y, Z \in \mathcal{V}$-**Cat**.*

Given the diagram

$$\begin{tikzcd} & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

if F is a saturated embedding then there is a choice of pullback (i.e. of A, U, and V fitting into the diagram below)

$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "V"'] & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

such that V is also a saturated embedding and $FU = GV$ strictly. Moreover, this square has the strict *universal property.*

Proof. Let the objects of A be the set $\coprod_{z \in Z} \left(F^{-1}(z) \times G^{-1}(z)\right)$. Let U and V be the restrictions to A of the projections p_X and p_Y.

□

We now make some observations that are relevant to the behavior of subobjects; we omit proofs that are trivial.

Proposition 8.11. *For $X, Y \in \mathcal{V}$-**Cat** and $F : X \to Y$ a $\mathcal{V}$-functor, there is $X' \in \mathcal{V}$-**Cat** and $F' : X \to X'$, $G : X' \to Y$ such that G is a saturated embedding and $F = GF'$ strictly. Furthermore, we can pick X' and $G : X' \to Y$ to satisfy the following property:*

If $F = G'F''$ strictly for some $F'' : X \to X''$ and $G' : X'' \to Y$ such that G' is a saturated embedding, then there exists a $H : X' \to X''$ such that $G = G'H$ strictly and H is a saturated embedding. X' is unique up to isomorphism of $\mathcal{V}$-categories, and G up to precomposition by such an isomorphism. We call G the image *of F.*

If $F : X \to Y$ is $\mathcal{V}$-f.f. then $F' : X \to X'$ above is part of a $\mathcal{V}$-equivalence $X \underset{\tilde{F}}{\overset{F'}{\rightleftarrows}} X'$.

Proof. Let the objects of X' consist of all the objects of Y which are $\mathcal{V}$-isomorphic to some object in the image of F. Then for $a, b \in X'$, $X'(a, b)$ is determined up to isomorphism by the requirement that $G : X' \to Y$ be $\mathcal{V}$-f.f. The inclusion on objects thus gives a saturated embedding $G : X' \to Y$. This determines F' via the requirement that, for each $x \in X$, we must set $F'x$ equal to the unique $x' \in X'$ with $Gx' = Fx$. Uniqueness of X' up to $\mathcal{V}$-**Cat**-isomorphism is clear.

If F is $\mathcal{V}$-f.f. then there is clearly always a (noncanonical) choice of $\mathcal{V}$-functor $\tilde{F} : X' \to X$

such that we have $X \underset{\tilde{F}}{\overset{F'}{\rightleftharpoons}} X'$.

$\square$

Proposition 8.12. *Let X, Y, Z be $\mathcal{V}$-categories fitting into the below diagram*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "H"'] & & Y \arrow[dl, "G"] \\ & Z & \end{tikzcd}$$

which commutes up to $\mathcal{V}$-isomorphism.

(a) If both G and H are $\mathcal{V}$-f.f. then so is F.

(b) If both G and H are saturated embeddings, then there is a unique F' which is $\mathcal{V}$-isomorphic to F such that F' is a saturated embedding and $GF = H$ strictly.

(c) If the diagram commutes strictly and both G and H are saturated embeddings, then F is also a saturated embedding.

Proposition 8.13. *Let X, Y, Z be $\mathcal{V}$-categories fitting into the below diagram*

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y \xrightarrow{H} Z$$

which commutes up to $\mathcal{V}$-isomorphism, i.e. HF is $\mathcal{V}$-isomorphic to HG.

(a) If H is $\mathcal{V}$-f.f. then F is $\mathcal{V}$-isomorphic to G.

(b) If the diagram commutes strictly and if H is an embedding then $F = G$ strictly.

As a first application, Proposition 8.12 allows us, in the definition of a subobject, to consider $\iota : A \to X$ and $\iota' : A' \to X$ to represent the same object iff there is an isomorphism of $\mathcal{V}$-categories $F : A \to A'$ such that $\iota = \iota' F$ *strictly.*

8.2.2 The poset of subobjects

Let X be a $\mathcal{V}$-category. We now define the category $\operatorname{Sub} X$ of subobjects of X as follows:

The objects of $\operatorname{Sub} X$ are the subobjects of X.

Given $\mathcal{V}$-functors $F : A \to B$ and $F' : A' \to B'$, we consider them equivalent when there is a $\mathcal{V}$-**Cat**-isomorphism $G : A \to A'$ witnessing the equivalence of $\iota : A \to X$ and $\iota' : A' \to X$ as subobjects of X, along with a $\mathcal{V}$-**Cat**-isomorphism $H : B \to B'$ witnessing the equivalence of $\theta : B \to X$ and $\theta' : B' \to X$ as subobjects of X, such that the diagram below commutes up to $\mathcal{V}$-isomorphism:

$$\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "\iota"] \arrow[dd, "G"'] & & B \arrow[dl, "\theta"'] \arrow[dd, "H"] \\
& I & \\
A' \arrow[ur, "\iota'"] \arrow[rr, "F'"'] & & B' \arrow[ul, "\theta'"']
\end{tikzcd}$$

Then given $\iota : A \to X$ and $\theta : B \to X$ representing subobjects $[\iota]$ and $[\theta]$ of X, a morphism between these two subobjects is given by an equivalence class $[F]$ of a $\mathcal{V}$-functor $F : A \to B$ such that ι is $\mathcal{V}$-isomorphic to θF.

In this situation, if we hold A and B fixed then by Proposition 8.12 we have a unique representative $F : A \to B$ which is a saturated embedding such that $\iota = \theta F$ strictly. This process is natural in A and B by uniqueness.

By the above, and by Proposition 8.13 we have that $\operatorname{Sub} X$ is in fact a poset category.

Let $F : X \to Y$. From the above observations and by Proposition 8.10 we get a functor

$F^* : \mathrm{Sub}\, Y \to \mathrm{Sub}\, X$ which acts on (representatives of) subobjects of X by pullback.

For $\iota : A \to X$ representing an object of $\mathrm{Sub}\, X$, we can factor $F\iota$ into HG where $G : A \to \exists_F A$ is some $\mathcal{V}$-functor and $H : \exists_F A \to Y$ is the image of $F\iota$. From the properties of the image and by Proposition 8.12 it follows that this process is well-defined on $\mathrm{Sub}\, X$ and functorial, yielding a functor $\exists_F : \mathrm{Sub}\, X \to \mathrm{Sub}\, Y$. It is easily checked that $\exists_F$ is left adjoint to F^*.

We can construct a right adjoint $\forall_F : \mathrm{Sub}\, X \to \mathrm{Sub}\, Y$ to F^* by hand. Given $\iota : A \to X$ and the reduction map $\pi : Y \to Y_0$, take the objects of $\forall_F A$ to be the set

$$\{y \in \pi^{-1}(y_0) \mid \text{if } \pi F x = y_0, \text{ then there is some } a \in A \text{ such that } \iota a = x\}.$$

That is, it is the set of all objects of Y which are $\mathcal{V}$-isomorphic to some object $y \in Y$ satisfying the condition that every x for which Fx is $\mathcal{V}$-isomorphic to y is also "in A", i.e. of the form ιa for some $a \in A$. The inclusion into Y on the objects of $\forall_F A$ extends to a saturated embedding $\forall_F \iota : \forall_F : A \to Y$.

Now let $\theta : B \to X$ represent another subobject in $\mathrm{Sub}\, X$. Then $\forall_F B$ has as objects the set $\{y \in \pi^{-1}(y_0) \mid \text{if } \pi F x = y_0, \text{ then there is some } b \in B \text{ such that } \theta b = x\}$.

If there is a (representative of a) morphism in $\mathrm{Sub}\, X$ given by $G : A \to B$ where G is a saturated embedding and $\iota = \theta G$ strictly, then by construction we have that the set of objects of $\forall_F A$ is a subset of the set of objects of $\forall_F B$, so that the inclusion extends to a saturated embedding $\forall_F G : \forall_F A \to \forall_F B$ such that $\forall_F \iota = (\forall_F \theta) G$ strictly. This process is well-defined and functorial on $\mathrm{Sub}\, X$, so that we have a functor $\forall_F : \mathrm{Sub}\, X \to \mathrm{Sub}\, Y$. As with $\exists_F$, it is easy to check that $\forall_F$ right adjoint to F^*.

We record these results as a proposition:

Proposition 8.14. *For $X, Y \in \mathcal{V}$-**Cat** and a $\mathcal{V}$-functor $F : X \to Y$, the functor $F^* : Sub\,Y \to Sub\,X$ has both left and right adjoints, given by $\exists_F$ and $\forall_F$, respectively.*

The lattice structure of $\mathrm{Sub}\,X$ for $X \in \mathcal{V}$-**Cat** shall not be explored here, for reasons of efficiency: meets and joins are constructed easily enough in analogy with Proposition 7.4, and implications, when they are relevant, may be constructed "by hand" in our cases of interest.

8.3 Set as a category of enriched categories

As before, we will use the category **Set** to guide our intuition. The rough idea is that, just as a collection of sets and functions between them serve as models of a given language of classical first order logic, with **Set** as the ambient category; for some appropriate choice $\mathcal{V}$ of enriching category we will regard a collection of $\mathcal{V}$-categories and morphisms between them ($\mathcal{V}$-functors) as potential models of a given language for a different kind of logic (thus the category $\mathcal{V}$-**Cat** of $\mathcal{V}$-categories will serve as the ambient category in which to interpret such a logic).

To properly ground our intuition, then, it is appropriate to regard **Set** as a *category of enriched categories*, i.e. to regard each set $X \in$ **Set** as a category enriched over an appropriate enriching category $\mathcal{V}$.

Remark 8.15. This basic idea of regarding **Set** as (a subcategory of) a category of enriched categories is a philosophically useful perspective for the project of interpreting different kinds of logic into categories. While the author has arrived independently at this idea (as well as the recognition of its usefulness as a perspective for our purposes) starting from [27], and while we were unable to find it in published literature, we are certainly not the first to conceive of it; for example the rough idea behind the material of this section is briefly discussed in the comments section of a post on the n-Category Café [2].

Let $\mathbb{2}$ denote the category $(\perp \to \top)$, i.e. the poset category with two objects (labeled by $\perp$ and $\top$) and $\perp \to \top$ the only non-identity morphism. There is a monoidal closed structure on $\mathbb{2}$, with tensor product (denoted by $\wedge$) given by $\top \wedge \top = \top$, and $a \wedge b = \perp$ for all other pairs (a, b). The monoidal identity is $\top$. We then have $\mathrm{Hom}_{\mathbb{2}}(a, b)$ (which we suggestively denote by $a \Rightarrow b$) given by $(\top \Rightarrow \perp) = \perp$ and $(a \Rightarrow b) = \top$ for all other pairs (a, b).

By $(\mathbb{2}\text{-}\mathbf{Cat})$ denote the category of κ-small $\mathbb{2}$-enriched categories, and by $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ denote the full subcategory of symmetric $\mathbb{2}$-enriched categories. We may think of $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ as the category of sets each equipped with an equivalence relation, and functions between them which respect equivalence classes. Two $\mathbb{2}$-functors $F, G : X \to Y$ are $\mathbb{2}$-isomorphic iff they are the same when regarded as functions between equivalence classes, or equivalently iff $\mathbb{2}\text{-}\mathbf{Cat}(X, Y)(F, G) = \top$. As is true for general $\mathcal{V}$, $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ is closed under the monoidal structure of $\mathbb{2}\text{-}\mathbf{Cat}$, so that $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ can itself be considered a monoidal category, with the inclusion $(\mathbb{2}\text{-}\mathbf{Cat})$-functor $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}} \to \mathbb{2}\text{-}\mathbf{Cat}$ strictly preserving the monoidal structure of $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$.

We can regard each $X \in \mathbf{Set}$ as a (symmetric) $\mathbb{2}$-category, by setting $X(a, a) = \top$ and $X(a, b) = \perp$ for $a \neq b$. This assignment takes the monoidal closed structure of $\mathbf{Set}$ to that of $\mathbb{2}\text{-}\mathbf{Cat}$; in particular there is a natural (in X and Y) bijection between the set of functions $\mathbf{Set}(X, Y)$ and the set of $\mathbb{2}$-functors between X and Y such that for two set functions $F, G : X \to Y$ regarded under this bijection as objects $F, G \in \mathbb{2}\text{-}\mathbf{Cat}(X, Y)$, F and G are the same as set functions iff we have $\mathbb{2}\text{-}\mathbf{Cat}(X, Y)(F, G) = \top$ (i.e. F and G are

$\mathbb{2}$-isomorphic). In this way we can consider **Set** to be a full subcategory of $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ with the inclusion $(\mathbb{2}\text{-}\mathbf{Cat})$-functor $\mathbf{Set} \to \mathbb{2}\text{-}\mathbf{Cat}$ strictly preserving the monoidal structure of **Set**.

Remark 8.16. From this perspective, we see that the definitions given in Section 8.1 reduce to the usual familiar notions:

(a) A $\mathbb{2}$-functor F between $X, Y \in \mathbf{Set}$ is just a set function $F : X \to Y$. F is $\mathbb{2}$-f.f. iff it is a saturated embedding iff it is an injective set function.

(b) For $X, Y \in \mathbf{Set}$, X is $\mathbb{2}$-equivalent to Y iff X is $(\mathbb{2}\text{-}\mathbf{Cat})$-isomorphic to Y iff X and Y are isomorphic as sets.

(c) If $\mathcal{A}$ and $\mathcal{B}$ are ordinary categories, i.e. categories enriched over **Set** (so in particular enriched over $\mathbb{2}$-**Cat**), then the notion of limits given in Definition 8.8 reduces to the usual notion of limits in a category. (And all limits have the strict universal property.)

We make the conceptual equivalence between $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ and **Set** precise:

Proposition 8.17. *There is a* $(\mathbb{2}\text{-}\mathbf{Cat})$*-functor* $Q : (\mathbb{2}\text{-}\mathbf{Cat})_{sym} \to \mathbf{Set}$ *such that for each* $X \in (\mathbb{2}\text{-}\mathbf{Cat})_{sym}$*, we have that* X *is* $\mathbb{2}$*-equivalent to* QX*.*

Proof. We first define Q on objects. Given $X \in (\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$, let QX be the $\mathbb{2}$-category with objects qx the $\mathbb{2}$-isomorphism classes of objects x in X. Let $QX(qx, qx) = \top$ and $QX(qx, qy) = \bot$ for $qx \neq qy$. QX is then an object of **Set**. (So QX is actually just the reduction X_0.)

Now given $X, Y \in (\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$, consider $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}(X, Y)$. There is clearly a $\mathbb{2}$-functor

$\bar{Q}_{X,Y} : (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(X,Y) \to (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(X,QY)$ given on objects by

$$(F : x \mapsto y) \mapsto (\bar{Q}_{X,Y}F : x \mapsto qy).$$

Now every object of $(\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(X,QY)$ maps each $\mathbb{2}$-isomorphism class in X to a single object in QY since each object of QY is its own $\mathbb{2}$-isomorphism class, and so descends to an object of $(\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(QX,QY)$, which gives an evident $\mathbb{2}$-functor

$$P_{X,Y} : (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(X,QY) \to (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}(QX,QY) = \mathbf{Set}(QX,QY).$$

We set $Q_{X,Y} = P_{X,Y}\bar{Q}_{X,Y}$. This construction is natural in X and Y, so that we have a $(\mathbb{2}\text{-}\mathbf{Cat})$-functor $Q : (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}} \to \mathbf{Set}$, as desired.

Given $X \in (\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}$, there is evidently a $\mathbb{2}$-f.f. $\mathbb{2}$-functor $Q_X : X \to QX$ that takes objects $x \in X$ to their $\mathbb{2}$-isomorphism class $qx \in QX$. Conversely, for each $a \in QX$ there is some $x_a \in X$ such that $a = qx_a$. Then $a \mapsto x_a$ becomes the object function of a $\mathbb{2}$-f.f. $\mathbb{2}$-functor $R_X : QX \to X$. We have that $Q_X R_X = 1_{QX} : QX \to QX$ and that $R_X Q_X : X \to X$ is $\mathbb{2}$-isomorphic to $1_X : X \to X$.

$\square$

The interpretation of classical logic into **Set** extends easily to an interpretation into $(\mathbb{2}\text{-}\mathbf{Cat})_{\text{sym}}$, and it is then straightforward to generalize to $\mathcal{V}$ to give an interpretation of "$\mathcal{V}$-valued" (in general intuitionistic) logic into $(\mathcal{V}\text{-}\mathbf{Cat})_{\text{sym}}$, using the machinery we have set up so far. However, since our goal is to give an interpretation of continuous logic, we will continue with developing the necessary categorical framework.

Chapter 9

Continuity in enriched categories

So far we have mostly adhered to the existing framework of enriched category theory, and exhibited **Set** as a kind of category of enriched categories, thereby motivating some intuition of what an interpretation of a logic into such a category should look like. Our goal is to exhibit the category of (extended pseudo-)metric spaces and uniformly continuous maps as such a "category of enriched categories", and then use this description to give a categorical interpretation of continuous logic. The notions introduced in this chapter are new (unless specifically mentioned otherwise), but are designed to yield properties analogous to those of the existing theory of interpreting logic into categories.

We must generalize the notion of a $\mathcal{V}$-functor, but first we need the following construction:

Definition 9.1. Given $\mathcal{V}$, let E be a κ-small collection of endofunctors $\epsilon : \mathcal{V} \to \mathcal{V}$ such that each ϵ is cocontinuous at I, i.e. whenever we have a diagram $D : J \to \mathcal{V}$ whose colimit is $\operatorname{colim} D \simeq I$, we have that $\epsilon(\operatorname{colim} D) \simeq I$ is the colimit of the diagram ϵD. In particular $\epsilon(I) \simeq I$.

Let E be closed under composition, i.e. if $\epsilon_1, \epsilon_2 \in E$ then $\epsilon_2 \circ \epsilon_1 \in E$, and also let E contain the identity endofunctor $1_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$.

Furthermore, for any $\epsilon_1, \epsilon_2 \in E$, let there be $\epsilon \in E$ such that for every $a, b \in \mathcal{V}$, there is a $\mathcal{V}$-morphism $\epsilon(a \otimes b) \to \epsilon_1(a) \otimes \epsilon_2(b)$. We say that ϵ *splits tensors* for ϵ_1 and ϵ_2.

We call E a *category of $\mathcal{V}$-moduli*, and each $\epsilon \in E$ a *$\mathcal{V}$-modulus*.

If, furthermore, every $\epsilon \in E$ is actually a monoidal functor $\mathcal{V} \to \mathcal{V}$ (so that ϵ also comes with a specified $\mathcal{V}$-morphism $\epsilon(r) \otimes \epsilon(s) \to \epsilon(r \otimes s)$ for each $r, s \in \mathcal{V}$), then we call E a *category of monoidal $\mathcal{V}$-moduli* and each ϵ a *monoidal $\mathcal{V}$-modulus*.

For example, $E_0 = \{1_{\mathcal{V}}\}$ is trivially a category of (monoidal) $\mathcal{V}$-moduli. There is also E_m, the maximal category of (not necessarily monoidal) $\mathcal{V}$-moduli, where for $\epsilon_1, \epsilon_2 \in E$ we have that $\epsilon = \epsilon_1 \otimes \epsilon_2$ splits tensors for ϵ_1 and ϵ_2.

When it is clear from context that E is a category of $\mathcal{V}$-moduli but we wish to emphasize that it is in fact a category of *monoidal* $\mathcal{V}$-moduli, we may simply say "E is monoidal".

Given $\mathcal{V}$-categories X and Y and a category of *monoidal* $\mathcal{V}$-moduli E, define a $(\mathcal{V}, E)$-functor (F, ϵ) to be an assigment of a Y-object Fa to each X-object a, along with a $\mathcal{V}$-morphism $(F, \epsilon)_{a,b} : \epsilon(X(a, b)) \to Y(Fa, Fb)$ for each pair of X-objects a, b such that the following diagrams commute:

$$\begin{tikzcd}
\epsilon(X(a,b)) \otimes \epsilon(X(b,c)) \arrow[r] \arrow[d, "(F,\epsilon)_{a,b} \otimes (F,\epsilon)_{b,c}"'] & \epsilon(X(a,b) \otimes X(b,c)) \arrow[r, "\epsilon(\mu^X_{a,b,c})"] & \epsilon(X(a,c)) \arrow[d, "(F,\epsilon)_{a,c}"] \\
Y(Fa,Fb) \otimes Y(Fb,Fc) \arrow[rr, "\mu^Y_{Fa,Fb,Fc}"] & & Y(Fa,Fc)
\end{tikzcd}$$

$$\begin{tikzcd}
 & \epsilon(X(a,a)) \arrow[d, "F_{a,a}"] \\
I \arrow[ur, "\epsilon(\eta^X_a)"] \arrow[dr, "\eta^Y_{Fa}"'] & \\
 & Y(Fa,Fa)
\end{tikzcd}$$

If $(F, \epsilon) : X \to Y$ and $(F, \epsilon') : X \to Y$ are equal on objects then we call them *essentially equal.*

Note that if $E = E_0$ then we recover the usual notion of a $\mathcal{V}$-functor.

We may refer to a $(\mathcal{V}, E)$-functor (F, ϵ) as simply F if the $\mathcal{V}$-modulus ϵ is either clear from context or unimportant.

Example 9.2. As in [27], let $\mathcal{V} = \mathbb{R}$ where by $\mathbb{R}$ we mean the monoidal closed poset category with objects given by the nonnegative real numbers, morphisms given by $a \to b$ iff $a \geq b$, and the tensor product given by addition. Then the monoidal unit is given by $I = 0$ and the internal hom by $\mathrm{Hom}_{\mathbb{R}}(a, b) = \max(b - a, 0)$.

If X is a symmetric $\mathbb{R}$-category (that is, for each $a, b \in X$ we have $X(a, b) = X(b, a)$), then the $\mathbb{R}$-category structure on X makes it into a metric space (where by "metric" we actually mean "extended pseudometric", meaning that distances between distinct points can be 0, and distances are allowed to be infinite).

Depending on our choice of E, the definition of $(\mathbb{R}, E)$-functors gives us different notions of maps between metric spaces. If $E = E_0$ then an $(\mathbb{R}, E_0)$-functor is just an ordinary $\mathbb{R}$-functor, giving a distance nonincreasing map between metric spaces. If E is instead the collection of all $\epsilon_\lambda : \mathbb{R} \to \mathbb{R}$ given by $r \mapsto \lambda r$ for each λ a nonnegative real number, then the collection of $(\mathbb{R}, E)$-functors is precisely the collection of Lipschitz maps.

The above example with $\mathcal{V} = \mathbb{R}$ already gives us a way to talk about Lipschitz maps between metric spaces as certain functors, but since ultimately our goal is to be able to talk about the still more general notion of uniformly continuous maps, we will need an even less restrictive notion of a $\mathcal{V}$-functor, which we define below. Recall that we are assuming that $\mathcal{V}$ is not only a cosmos with monoidal unit given by the terminal object, but also a poset category.

Definition 9.3. Let E be a category of (not necessarily monoidal) $\mathcal{V}$-moduli.

Given $\mathcal{V}$-categories X and Y, define a *loose* $(\mathcal{V}, E)$*-functor* $(F, \epsilon) : X \to Y$ to be an assigment of a Y-object Fa to each X-object a, along with a $\mathcal{V}$-morphism $(F, \epsilon)_{a,b} : \epsilon(X(a, b)) \to Y(Fa, Fb)$ for each pair of X-objects a, b such that the following diagram commutes:

$$\begin{tikzcd} & \epsilon(X(a,a)) \arrow[dd, "F_{a,a}"] \\ I \arrow[ur, "\epsilon(\eta^X_a)"] \arrow[dr, "\eta^Y_{Fa}"'] & \\ & Y(Fa, Fa) \end{tikzcd}$$

The composition of two loose $(\mathcal{V}, E)$-functors is again a loose $(\mathcal{V}, E)$-functor, since E is closed under composition. Thus we denote the κ'-small category with objects κ-small $\mathcal{V}$-categories and morphisms loose $(\mathcal{V}, E)$-functors by $(\mathcal{V}, E)$-**Cat**.

As in the non-loose case, we call two loose $(\mathcal{V}, E)$-functors $(F, \epsilon) : X \to Y$ and $(F, \epsilon') : X \to Y$ *essentially equal* when they are equal on objects.

Note that a loose $(\mathcal{V}, E)$-functor is just the same as a $(\mathcal{V}, E)$-functor where we have discarded the functoriality (!) conditions. We will see that it is still possible to develop a coherent theory in such a relaxed setting, in large part because $\mathcal{V}$ satisfies quite strong conditions and because the categorical structure we are interested in is that of $(\mathcal{V}, E)$-**Cat** and not so much that of individual $\mathcal{V}$-categories themselves.

Indeed, in the case that $\mathcal{V} = \mathbb{R}$ and $E = E_m$, $(\mathbb{R}, E_m)$-**Cat** will essentially be the category $(\mathbf{pMet}_\infty)_u$, the category of metric spaces and uniformly continuous maps, so that a $(\mathbb{R}, E_m)$-functor $(F, \epsilon) : X \to Y$ may be thought of as a uniformly continuous map F with modulus of uniform continuity ϵ. Morphism composition in an $\mathbb{R}$-category simply amounts to witnessing the triangle inequality, so the abandonment of the functoriality condition for loose $\mathbb{R}$-functors corresponds in this case to the fact that the triangle inequality in the domain of a uniformly continuous map is not what guarantees the triangle inequality in the image; the triangle inequality is separately enforced on the codomain simply by virtue of the codomain being a metric space. That the notion of a uniformly continuous map may be carried over wholesale to the setting of *semi*metric spaces (i.e. ones in which the triangle inequality does not hold) suggests that we in fact should not expect a categorical formulation of "uniformly continuous map" to be compatible with rigid functoriality properties.

The reason, then, that we expend the effort of constructing $(\mathcal{V}, E)$-**Cat** only to later specialize to the case of an already known ordinary category is that the morally 2-categorical perspective of treating $(\mathcal{V}, E)$-**Cat** as a category of (enriched) categories naturally yields the correct formulations of the various constructions we will need. For example, we will need to speak of isometric embeddings, but objects $(\mathbf{pMet}_\infty)_u$ are considered isomorphic when they are uniformly homeomorphic, which is too coarse a notion of isomorphism. (From the $(\mathcal{V}, E)$-**Cat** perspective, such objects will be considered equivalent but not necessarily isomorphic.)

9.1 Some features of $(\mathcal{V}, E)$-Cat

We now give the "loose" versions of the definitions and results that we will need, adapted from Chapter 8. Some notions, for example what it means for a $\mathcal{V}$-category to be symmetric or for two objects a, b of a $\mathcal{V}$-category X to be $\mathcal{V}$-isomorphic, do not reference the notion of a functor and therefore remain entirely unchanged. In all that follows, when we say "$(\mathcal{V}, E)$-functor" we will mean "loose $(\mathcal{V}, E)$-functor".

Definition 9.4. Let E be a category of $\mathcal{V}$-moduli, $\epsilon \in E$ a $\mathcal{V}$-modulus, and X, Y $\mathcal{V}$-categories.

Let $(F, \epsilon) : X \to Y$ be a $(\mathcal{V}, E)$-functor.

(a) We say that (F, ϵ) is *$\mathcal{V}$-f.f.* when there is an essentially equal $(F, 1_{\mathcal{V}}) : X \to Y$ such that $(F, 1_{\mathcal{V}})_{a,b} : X(a, b) \to Y(Fa, Fb)$ is an isomorphism in $\mathcal{V}$ for each $a, b \in X$.

(b) If in addition to being $\mathcal{V}$-f.f. F is also injective on objects then we call it an *embedding of $\mathcal{V}$-categories.*

(c) If in addition to being $\mathcal{V}$-f.f. we have that F is a bijection on objects, then we call it an *isomorphism of $\mathcal{V}$-categories.*

(d) We say that (F, ϵ) is *saturated* when for every $x \in X$ and every $y \in Y$ such that y is $\mathcal{V}$-isomorphic to Fx in Y, there is some $x' \in X$ such that $Fx' = y$.

9.1.1 A notion of enrichment for $(\mathcal{V}, E)$-Cat

As is the case with $\mathcal{V}$-**Cat**, for each $X, Y \in (\mathcal{V}, E)$-**Cat** we can give the set $((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Y)$ the structure of a $\mathcal{V}$-category by exactly the same process. Namely, for

$F, G \in ((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Y)$, we let $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)(F, G) = \prod_a Y(Fa, Ga)$.

Also as is the case with $\mathcal{V}$-**Cat**, we can specify a tensor product on $(\mathcal{V}, E)$-**Cat** in the same way: for $X, Y \in (\mathcal{V}, E)$-**Cat** we define $X \otimes Y$ to be the $\mathcal{V}$-category with objects (a, b) where $a \in X$ and $b \in Y$, and $(X \otimes Y)((a_1, b_1), (a_2, b_2)) = X(a_1, a_2) \otimes Y(b_1, b_2)$. The monoidal unit is the same as with $\mathcal{V}$-**Cat**, i.e. it is the category $\mathbb{I}$ with one object $*$ where $\mathbb{I}(*, *) = I$.

This allows us to say that two $(\mathcal{V}, E)$-functors $(F, \epsilon), (G, \eta) : X \to Y$ are $\mathcal{V}$-isomorphic when they are $\mathcal{V}$-isomorphic as objects of $(\mathcal{V}, E)$-**Cat**(X, Y). Note that the relation of being $\mathcal{V}$-isomorphic does not depend on $\mathcal{V}$-moduli. Relatedly, if (F, ϵ) and (G, η) are $\mathcal{V}$-isomorphic then we can take η to be a $\mathcal{V}$-modulus for F and ϵ to be a $\mathcal{V}$-modulus of G, so that we have that (F, η) and (G, ϵ) are $(\mathcal{V}, E)$-functors. Also note that two functors which are essentially equal are automatically $\mathcal{V}$-isomorphic.

Moreover, for a given $\epsilon \in E$ and $X, Y \in (\mathcal{V}, E)$-**Cat** we can speak of the "full subcategory" $((\mathcal{V}, E)\text{-}\mathbf{Cat})_\epsilon(X, Y)$, whose set of objects is given by the set of all $(\mathcal{V}, E)$-functors $(F, \epsilon') : X \to Y$ such that there exists a natural transformation $\epsilon \to \epsilon'$ in the (ordinary) functor category $[\mathcal{V}, \mathcal{V}]$, i.e. for all $a \in \mathcal{V}$ there is a morphism $\epsilon(a) \to \epsilon'(a)$. For $F, G \in ((\mathcal{V}, E)\text{-}\mathbf{Cat})_\epsilon(X, Y)$ we set $((\mathcal{V}, E)\text{-}\mathbf{Cat})_\epsilon(X, Y)(F, G) = (\mathcal{V}, E)\text{-}\mathbf{Cat}(F, G)$.

Theorem 9.5. *Let E be a category of $\mathcal{V}$-moduli, and $X, Y, Z \in (\mathcal{V}, E)$-**Cat**. Fix some $\epsilon' \in E$.*

The set map $((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(Y, Z) \times ((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Y) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Z)$ given by composition of $(\mathcal{V}, E)$-functors restricts to the underlying object function of a $(\mathcal{V}, E)$-functor $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon'}(Y, Z) \otimes (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y) \to (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Z)$.

Proof. Let $(F_1, \epsilon_{F_1}), (F_2, \epsilon_{F_2}) \in (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)$ and $(G_1, \epsilon_{G_1}), (G_2, \epsilon_{G_2}) \in ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon'}(Y, Z)$. For simplicity denote $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)$ by Ω_1, $((\mathcal{V}, E)\text{-}\mathbf{Cat}(Y, Z))_{\epsilon'}$ by Ω_2, and $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Z)$

by Ω. We wish to give a $\mathcal{V}$-morphism

$$\epsilon\left(\Omega_2 \otimes \Omega_1((F_1, G_1), (F_2, G_2))\right) = \epsilon\left(\Omega_2(G_1, G_2) \otimes \Omega_1(F_1, F_2)\right) \to \Omega(G_1 F_1, G_2 F_2)$$

for some $\epsilon \in E$.

This means that we need to find some $\epsilon \in E$ and a morphism

$$\epsilon\left(\left(\prod_y Z(G_1 y, G_2 y)\right) \otimes \left(\prod_x Y(F_1 x, F_2 x)\right)\right) \to \prod_x Z(G_1 F_1 x, G_2 F_2 x)$$

in $\mathcal{V}$.

Let $\epsilon \in E$ be a $\mathcal{V}$-modulus that splits tensors for $1_{\mathcal{V}}$ and ϵ', so that we have

$$\epsilon\left(\left(\prod_y Z(G_1 y, G_2 y)\right) \otimes \left(\prod_x Y(F_1 x, F_2 x)\right)\right) \to \prod_y Z(G_1 y, G_2 y) \otimes \epsilon'\left(\prod_x Y(F_1 x, F_2 x)\right)$$

Let Y' denote the set of objects of Y that are *not* of the form $F_2 x$ for any $x \in X$, and let X' denote a set of objects of X on which F_2 is a bijection onto its whole image. Then we have that $\prod_y Z(G_1 y, G_2 y) = \left(\prod_{y \in Y'} Z(G_1 y, G_2 y)\right) \times \left(\prod_{x \in X'} Z(G_1 F_2 x, G_2 F_2 x)\right)$. However, since $\mathcal{V}$ is a poset we have that for any $a \in \mathcal{V}$, $a = \prod_{i \in J} a$ for any nonempty index set J, so that $\prod_{x \in X'} Z(G_1 F_2 x, G_2 F_2 x) = \prod_x Z(G_1 F_2 x, G_2 F_2 x)$.

Then we have the following morphisms in $\mathcal{V}$:

$$\prod_y Z(G_1y, G_2y) = \left(\prod_{y\in Y'} Z(G_1y, G_2y)\right) \times \left(\prod_x Z(G_1F_2x, G_2F_2x)\right)$$
$$\xrightarrow{\pi_2} \prod_x Z(G_1F_2x, G_2F_2x)$$
$$\epsilon'\left(\prod_x Y(F_1x, F_2x)\right) \to \epsilon_{G_1}\left(\prod_x Y(F_1x, F_2x)\right) \to \prod_x \epsilon_{G_1}(Y(F_1x, F_2x))$$
$$\xrightarrow{\prod_x (G_1)_{F_1x,F_2x}} \prod_x Z(G_1F_1x, G_1F_2x)$$

Now for each $x \in X$ we have a map

$$\prod_x Z(G_1F_2x, G_2F_2x) \otimes \prod_x Z(G_1F_1x, G_1F_2x) \xrightarrow{\pi_x\otimes\pi_x} Z(G_1F_2x, G_2F_2x) \otimes Z(G_1F_1x, G_1F_2x)$$
$$\xrightarrow{\mu^Z} Z(G_1F_1x, G_2F_2x)$$

which gives us a morphism

$$\prod_x Z(G_1F_2x, G_2F_2x) \otimes \prod_x Z(G_1F_1x, G_1F_2x) \to \prod_x Z(G_1F_1x, G_2F_2x) = \Omega(G_1F_1, G_2F_2)$$

Putting all of the above together gives us the required $\epsilon \in E$ and a $\mathcal{V}$-morphism

$$\epsilon(((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon'}(Y, Z) \otimes (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)((F_1, G_1), (F_2, G_2)))$$
$$\to (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Z)(G_1F_1, G_2F_2)$$

for each $F_1, F_2 \in (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)$ and $G_1, G_2 \in ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon'}(Y, Z)$.

$\mathcal{V}$-functoriality is immediate since $\mathcal{V}$ is a poset, so that we have a $\mathcal{V}$-functor

$$((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon'}(Y, Z) \otimes (\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y) \to \mathcal{V}\text{-}\mathbf{Cat}(X, Z)$$

which agrees at the object level with (the restriction of) the set function

$$((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(Y, Z) \times ((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Y) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_0(X, Z)$$

□

The fact that for $(F, \epsilon) : X \to Y$ and $(G, \eta) : Y \to Z$ we have $(GF, \eta\epsilon) : X \to Z$ gives us, via the above, that "composition of $(\mathcal{V}, E)$-functors" yields a $(\mathcal{V}, E)$-functor $((\mathcal{V}, E)\text{-}\mathbf{Cat})_\eta(Y, Z) \otimes ((\mathcal{V}, E)\text{-}\mathbf{Cat})_\epsilon(X, Y) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\eta\epsilon}(X, Z)$, with some $\mathcal{V}$-modulus $\zeta \in E$.

Now if for $\epsilon_1, \epsilon_2, \eta_1, \eta_2 \in E$ we have $\epsilon_2 \to \epsilon_1$ and $\eta_2 \to \eta_1$ in $[\mathcal{V}, \mathcal{V}]$, then for $X, Y, Z \in (\mathcal{V}, E)\text{-}\mathbf{Cat}$ we have embeddings of $\mathcal{V}$-categories

$$((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\eta_1}(Y, Z) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\eta_2}(Y, Z)$$

and

$$((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon_1}(X, Y) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon_2}(X, Y).$$

Clearly the composition $(\mathcal{V}, E)$-functor

$$((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\eta_1}(Y, Z) \otimes ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\epsilon_1}(X, Y) \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\eta_1\epsilon_1}(X, Z)$$

agrees with the restriction of the composition functor

$$((\mathcal{V},E)\text{-}\mathbf{Cat})_{\eta_2}(Y,Z)\otimes((\mathcal{V},E)\text{-}\mathbf{Cat})_{\epsilon_2}(X,Y)\to((\mathcal{V},E)\text{-}\mathbf{Cat})_{\eta_2\epsilon_2}(X,Z).$$

This suggests a way we can consider $(\mathcal{V},E)$-**Cat** to be enriched over itself. As we have seen, E has a poset structure inherited from the poset structure of $[\mathcal{V},\mathcal{V}]$, and the above discussion shows that for each $X,Y\in(\mathcal{V},E)$-**Cat** there is a functor $\mathcal{W}_{a,b}:E^{\mathrm{op}}\to(\mathcal{V},E)$-**Cat** given by $\epsilon\mapsto((\mathcal{V},E)\text{-}\mathbf{Cat})_{\epsilon}(X,Y)$. Furthermore, there is an associative composition operation $\mathrm{comp}:E^{\mathrm{op}}\times E^{\mathrm{op}}\to E^{\mathrm{op}}$ given by $(\eta,\epsilon)\mapsto\eta\epsilon$ that respects the poset structure of E, i.e. a functor $E^{\mathrm{op}}\times E^{\mathrm{op}}\to E$ between poset categories. For technical reasons, we consider E', the full subcategory of E with only the objects ϵ for which there exists an arrow $\epsilon\to 1_{\mathcal{V}}$. (Then $(E')^{\mathrm{op}}$ is the full subcategory of E^{op} with only those ϵ for which there exists an arrow $1_{\mathcal{V}}\to\epsilon$.) Since $\mathrm{comp}:E^{\mathrm{op}}\times E^{\mathrm{op}}\to E^{\mathrm{op}}$ is in particular a poset map it restricts to a map $\mathrm{comp}:(E')^{\mathrm{op}}\times(E')^{\mathrm{op}}\to(E')^{\mathrm{op}}$.

Definition 9.6. Let $\mathcal{C}$ be a category equipped with a bifunctor $\mathrm{comp}:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which is associative, i.e. satisfies $\mathrm{comp}\circ(1_{\mathcal{C}}\times\mathrm{comp})=\mathrm{comp}\circ(\mathrm{comp}\times 1_{\mathcal{C}})$.

Let $\mathcal{W}$ be a monoidal category equipped with an equivalence relation $\sim$ on each of its homsets. Then for any category $\mathcal{B}$, there is an induced equivalence relation $\sim$ on morphisms of the functor category $[\mathcal{B},\mathcal{W}]$ given by $\mu\sim\nu$ whenever $\mu(b)\simeq\nu(b)$ for all $b\in\mathcal{B}$. Denote the monoidal unit of $\mathcal{W}$ by I, and for $\mathcal{B}$ a category denote by $I_{\mathcal{B}}$ the functor constant at I.

We say that $\mathcal{A}$ is a *$\mathcal{C}$-indexed $\mathcal{W}$-category* (or $\mathcal{W}(\mathcal{C})$-category, for short) when we are given the following data:

(a) A set $\mathcal{A}_0$ of *objects*;

(b) For each pair a, b of objects, a functor $\mathcal{W}_{a,b} : \mathcal{C} \to \mathcal{W}$;

(c) For each triple a, b, c of objects, a natural transformation

$$\mu_{a,b,c} : \otimes \circ (\mathcal{W}_{b,c} \times \mathcal{W}_{a,b}) \to \mathcal{W}_{a,c} \circ \mathrm{comp}$$

(i.e. a morphism in the functor category $[\mathcal{C} \times \mathcal{C}, \mathcal{W}]$) that satisfies the following associativity condition:

For $a, b, c, d \in \mathcal{A}_0$, the composite natural transformation

$$(\mu_{a,c,d} \circ (1_{\mathcal{C}} \times \mathrm{comp})) \cdot (\otimes \circ (1_{\mathcal{W}_{c,d}} \times \mu_{a,b,c}))$$

is equivalent under $\sim$ to the composite

$$(\mu_{a,b,d} \circ (\mathrm{comp} \times 1_{\mathcal{C}})) \cdot (\otimes \circ (\mu_{b,c,d} \times 1_{\mathcal{W}_{a,b}}))$$

(modulo the canonical associativity isomorphisms in $\mathcal{W}$ and **Cat**).

(d) For each $a \in \mathcal{A}_0$, a natural transformation $\eta_a^{\mathcal{A}} : I_{\mathcal{C}} \to \mathcal{W}_{a,a}$ satisfying the necessary unital conditions.

Given $\mathcal{W}(\mathcal{C})$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{W}(\mathcal{C})$-*functor* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between them is given by the following data:

(a) A functor $M : \mathcal{C} \to \mathcal{C}$ and a natural transformation $m : M \circ \mathrm{comp} \to \mathrm{comp} \circ (M \times M)$ in $[\mathcal{C} \times \mathcal{C}, \mathcal{C}]$.

(b) For each $a, b \in \mathcal{A}$, a natural transformation $\mathcal{F}_{a,b} : \mathcal{W}_{a,b} \to \mathcal{W}_{\mathcal{F}a,\mathcal{F}b} \circ M$ satisfying, for each $a, b, c \in \mathcal{A}$, the evident functoriality conditions with respect to $\mu^{\mathcal{A}}_{a,b,c}$ and $\mu^{\mathcal{B}}_{\mathcal{F}a,\mathcal{F}b,\mathcal{F}c}$.

(c) For each $a \in \mathcal{A}$, $\mathcal{F}_{a,a} \cdot \eta^{\mathcal{A}}_a \sim \eta^{\mathcal{B}}_{\mathcal{F}a} \circ M$.

Note that if we set $\mathcal{C}$ equal to the terminal category (with one object and one identity morphism), then we recover the usual notion of a $\mathcal{W}$-enriched category and $\mathcal{W}$-functors between them.

Then we set $\mathcal{C} = (E')^{\text{op}}$; $\mathcal{W} = (\mathcal{V}, E)$-**Cat**; and $\sim$ to be the relation of $(\mathcal{V}, E)$-functors being essentially equal. For $X, Y \in (\mathcal{V}, E)$-**Cat** we let $\mathcal{W}_{X,Y}$ be as in our discussion above, and for each $\epsilon \in (E')^{\text{op}}$ and $X, Y, Z \in (\mathcal{V}, E)$-**Cat** we have that

$\mu_{X,Y,Z} : \mathcal{W}_{Y,Z} \otimes \mathcal{W}_{X,Y} \to \mathcal{W}_{X,Z}$ is just the $(\mathcal{V}, E)$-functor giving composition in $(\mathcal{V}, E)$-**Cat**. Then for each $\epsilon \in (E')^{\text{op}}$, $\eta^{\mathcal{A}}_X(\epsilon)$ is given by the $(\mathcal{V}, E)$-functor $\mathbb{I} \to ((\mathcal{V}, E)\text{-}\mathbf{Cat})_\epsilon(X, X)$ sending $* \mapsto 1_{\mathcal{V}}$.

Thus $(\mathcal{V}, E)$-**Cat** is a $(E')^{\text{op}}$-indexed $((\mathcal{V}, E)$-**Cat**$)$-enriched category. Although in general $(E')^{\text{op}}$ does not contain the supremum (colimit) over all of its objects, it is clear that for each $X, Y \in (\mathcal{V}, E)$-**Cat** we have that the colimit $\text{colim}(\mathcal{W}_{X,Y})_0$ exists in **Set**, and is in fact the underlying set of $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)$; this gives us a way in which to consider $(\mathcal{V}, E)$-**Cat** as morally a $((\mathcal{V}, E)$-**Cat**$)$-enriched category.

Denote by $\mathcal{C} = (E')^{\text{op}}$ and consider the above setup. For $\mathcal{V}$-categories X and Y, $\epsilon, \epsilon' \in \mathcal{C}$, and $F \in \mathcal{W}_{X,Y}(\epsilon)$ and $G \in \mathcal{W}_{X,Y}(\epsilon')$, there is an $\epsilon'' \in \mathcal{C}$ with $i : \epsilon \to \epsilon''$ and $i' : \epsilon' \to \epsilon''$ (by the tensor splitting property of E, which makes $\mathcal{C}$ a directed poset) such that $\mathcal{W}_{X,Y}(i)(F) \in \mathcal{W}_{X,Y}(\epsilon'')$ is essentially equal to $F \in \mathcal{W}_{X,Y}(\epsilon)$ as elements of $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, Y)$ and $\mathcal{W}_{X,Y}(i')(G) \in \mathcal{W}_{X,Y}(\epsilon'')$ is essentially equal to $G \in \mathcal{W}_{X,Y}(\epsilon')$. Thus we say that F and G are $\mathcal{V}$-isomorphic (resp. essentially equal) "as elements of

$(\mathcal{V}, E)$-**Cat**(X, Y)" if $\mathcal{W}_{X,Y}(i)(F)$ and $\mathcal{W}_{X,Y}(i')(G)$ are as elements of $\mathcal{W}_{X,Y}(\epsilon'')$.

Henceforth, for any $\mathcal{A}$ which we declare to be a $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-category, for $X, Y \in \mathcal{A}$ and $\epsilon \in \mathcal{C}$ we denote $\mathcal{W}_{X,Y}(\epsilon)$ by $\mathcal{A}_\epsilon(X, Y)$, and we make the assumption that for all $X, Y \in \mathcal{A}$ we have that the colimit $\mathrm{colim}_{\mathcal{C}}(\mathcal{A}_\epsilon(X, Y))$ in **Set** is the underlying set of some $\mathcal{V}$-category which we denote by $\mathcal{A}(X, Y)$. This is natural, in the following sense: for $Z \in \mathcal{A}$ we have that the underlying set of $\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y)$ is

$$\mathrm{colim}_{\mathcal{C}}(\mathcal{A}_\epsilon(Y, Z))_0 \times \mathrm{colim}_{\mathcal{C}}(\mathcal{A}_\epsilon(X, Y))_0 \simeq \mathrm{colim}_{\mathcal{C}}\left((\mathcal{A}_\epsilon(Y, Z))_0 \times (\mathcal{A}_\epsilon(X, Y))_0\right)$$

(since $\mathcal{C}$ is directed and directed colimits commute with finite limits).

Furthermore, we require that for each $i : \epsilon \to \epsilon'$ in $\mathcal{C}$, $i_* : \mathcal{A}_\epsilon(X, Y) \to \mathcal{A}_{\epsilon'}(X, Y)$ is an embedding, which is certainly the case when $\mathcal{A} = (\mathcal{V}, E)$-**Cat**.

Note that all of the above can be repeated verbatim if we replace every occurrence of $(\mathcal{V}, E)$-**Cat** above with $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$, the full subcategory of $(\mathcal{V}, E)$-**Cat** with objects only the symmetric $\mathcal{V}$-categories. That is, $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$ is a $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-category, and the inclusion on objects extends to a $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-functor from $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$ to $(\mathcal{V}, E)$-**Cat**; on the other hand there is clearly a $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}(\mathcal{C})$-enrichment of $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$. These enrichments are compatible in the sense that the $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-enrichment of $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$ naturally restricts along the inclusion to give the $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}(\mathcal{C})$-enrichment of $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$.

We have constructed the above notion of a "$\mathcal{C}$-indexed" enrichment to enable us to speak of diagrams in (functors into) $(\mathcal{V}, E)$-**Cat** and $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$. However, when we are working within $(\mathcal{V}, E)$-**Cat** and $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$, it is mostly harmless to forget the $\mathcal{C}$-indexing; when this is the case we will do so to avoid notation overload.

We now continue to give the "loose" adaptations of the features of $\mathcal{V}$-**Cat**. For the most part the adaptations are straightforward; for the sake of efficiency we will cover only those notions directly relevant to our objective of interpreting continuous logic into $(\mathbb{R}, E_m)$-**Cat**. Furthermore, unless otherwise stated, every construction and result below for $(\mathcal{V}, E)$-**Cat** should be understood as also applying to $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$, with the same proofs.

Definition 9.7. Given $X, Y \in (\mathcal{V}, E)$-**Cat**, we say that X, Y are $(\mathcal{V}, E)$-*equivalent* when there are $(\mathcal{V}, E)$-functors $F : X \to Y$ and $G : Y \to X$ such that GF is $\mathcal{V}$-isomorphic to 1_X in $(\mathcal{V}, E)\text{-}\mathbf{Cat}(X, X)$ and FG is $\mathcal{V}$-isomorphic to 1_Y in $(\mathcal{V}, E)\text{-}\mathbf{Cat}(Y, Y)$. We may denote this situation as $X \underset{G}{\overset{F}{\rightleftharpoons}} Y$.

If furthermore we have that GF is essentially equal to 1_X and FG is essentially equal to 1_Y then we say that X and Y are *strongly* $(\mathcal{V}, E)$-*equivalent*, and denote this as $X \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftharpoons}} Y$.

As for $\mathcal{V}$-**Cat**, we adopt the following convention, that when we say two $(\mathcal{V}, E)$-functors $G : X \to Y$ and $G' : X' \to Y'$ are $(\mathcal{V}, E)$-*equivalent*, we mean that there is a $(\mathcal{V}, E)$-equivalence $X \underset{F'}{\overset{F}{\rightleftharpoons}} X'$ and a $(\mathcal{V}, E)$-equivalence $Y \underset{H'}{\overset{H}{\rightleftharpoons}} Y'$ such that G' is $\mathcal{V}$-isomorphic to HGF' ($\Leftrightarrow G$ is $\mathcal{V}$-isomorphic to $H'G'F \Leftrightarrow HG$ is $\mathcal{V}$-isomorphic to $G'F \Leftrightarrow GF'$ is $\mathcal{V}$-isomorphic to $H'G'$). If we take $F = F' = 1_X$ then we say that this $(\mathcal{V}, E)$-equivalence of functors *fixes* X, while if $G = G' = 1_Y$ then we say that the $(\mathcal{V}, E)$-equivalence *fixes* Y.

We repeat the above with all $(\mathcal{V}, E)$-equivalences replaced with strong $(\mathcal{V}, E)$-equivalences and "$\mathcal{V}$-isomorphic" with "essentially equal" to say when G and G' are strongly $(\mathcal{V}, E)$-equivalent.

Because the notion of subobject is central, we restate the definition of a subobject below. We omit mentions of $\mathcal{V}$-moduli below for convenience, and because all the $(\mathcal{V}, E)$-functors in question can be taken to have $\mathcal{V}$-modulus $1_{\mathcal{V}}$.

Definition 9.8. Let X be a $\mathcal{V}$-enriched category. By a *subobject* of X we mean the equivalence class of a saturated embedding $\iota : A \to X$, where $\iota : A \to X$ is considered equivalent to $\iota' : A' \to X$ when there is an isomorphism of $\mathcal{V}$-categories given by $F : A \to A'$ such that ι is $\mathcal{V}$-isomorphic to $\iota' F$.

We will often refer to a subobject of X by (the domain of) one of its representatives, i.e. "$\iota : A \to X$ is a subobject of X" or "A is a subobject of X" when no confusion will result. As we have done implicitly, we will often neglect to mention $\mathcal{V}$-moduli for these $(\mathcal{V}, E)$-functors , since we can always take them to have $\mathcal{V}$-modulus $1_{\mathcal{V}}$.

Definition 9.9. Let $\mathcal{A}, \mathcal{B}$ be $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-categories, and let $D : \mathcal{A} \to \mathcal{B}$ be a $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-functor, which we call a diagram.

By a *cone* λ *over* D we mean an object $X \in \mathcal{B}$ along with, for each $A \in \mathcal{A}$, an $\epsilon_A \in \mathcal{C}$ and an object $\lambda_A \in \mathcal{B}_{\epsilon_A}(X, DA)$ such that for every $F \in \mathcal{A}(A_1, A_2)$, we have that $(D_{A_1,A_2} F)\lambda_{A_1}$ is $\mathcal{V}$-isomorphic to λ_{A_2} in $\mathcal{B}(X, DA_2)$. We may also call λ a *cone from* X *to* D.

By a *limit of* D we mean an object $\lim D \in \mathcal{B}$ along with a cone λ from $\lim D$ to D such that for any $X \in \mathcal{B}$ and a cone μ from X to D, there is some $L \in \mathcal{B}(X, \lim D)$, unique up to $\mathcal{V}$-isomorphism, such that for each $A \in \mathcal{A}$, we have that μ_A is $\mathcal{V}$-isomorphic to $\lambda_A L$ in $\mathcal{B}(X, DA)$. Note that $\lim D$ is unique up to $(\mathcal{V}, E)$-equivalence.

If $\mathcal{B} = (\mathcal{V}, E)\text{-}\mathbf{Cat}$ and there is a choice of $\lim D$ along with a choice of limit cone λ satisfying the commutativity conditions up to essential equality and not only up to $\mathcal{V}$-isomorphism; and further if for each $X \in \mathcal{B}$ with a cone μ from X to D there exists an $L \in \mathcal{B}(X, \lim D)$ for each $A \in \mathcal{A}$ we have that μ_A is essentially equal to $\lambda_A L$, then we say that this choice of $\lim D$ has the *strict universal property.* Such a choice of $\lim D$ is unique up to strong $(\mathcal{V}, E)$-equivalence.

Cocones and colimits are defined dually.

Furthermore, if $\mathcal{A}$ is the empty $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-category and $\mathcal{B} = (\mathcal{V}, E)\text{-}\mathbf{Cat}$ then we can take $\lim D = \mathbb{I}$; thus the monoidal unit of $(\mathcal{V}, E)\text{-}\mathbf{Cat}$ is also its *terminal object*.

If $\mathcal{A}$ is taken to be the $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-category with three objects A_1, A_2, A_3 with $\mathcal{A}(A_1, A_1) = \mathcal{A}(A_2, A_2) = \mathcal{A}(A_3, A_3) = \mathcal{A}(A_1, A_3) = \mathcal{A}(A_2, A_3) = \mathbb{I}$ and empty hom-objects otherwise, then $\lim D$ is also called a *pullback*. ($\mathcal{A}$ is illustrated by the diagram below, where we have omitted the "identity $\mathcal{V}$-functors".)

$$\begin{tikzcd} & A_1 \arrow[d] \\ A_2 \arrow[r] & A_3 \end{tikzcd}$$

Proposition 9.10. $(\mathcal{V}, E)$-**Cat** *has pullbacks.*

Proof. The proof is essentially the same as for $\mathcal{V}$-**Cat**.

Let $X, Y, Z \in (\mathcal{V}, E)\text{-}\mathbf{Cat}$ and let $F : X \to Z$ and $G : Y \to Z$ be $(\mathcal{V}, E)$-functors fitting into the below diagram:

$$\begin{tikzcd} & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

We construct a $\mathcal{V}$-category A and $(\mathcal{V}, E)$-functors $(U, \epsilon) : A \to X$ and $(V, \eta) : A \to Y$ that satisfy the conditions for being a limit of the above diagram.

First note that if $Z = \mathbb{I}$ then we can take A to be the product $X \times Y$, which has objects pairs of the form (x,y) for $x \in X$ and $y \in Y$, with $X \times Y((x,y),(x',y')) = X(x,x') \times Y(y,y')$ (where the product is taken in $\mathcal{V}$). Then U and V are just the obvious projections p_X and p_Y.

Let Z_0 denote the reduction of Z and $\pi : Z \to Z_0$ the reduction map.

The set of objects of A is given by the set $\coprod_{z \in Z_0} \big((\pi F)^{-1}(z) \times (\pi G)^{-1}(z)\big)$. This is clearly a subset of the set of objects of $X \times Y$, so that every $a \in A$ is uniquely of the form (x,y) for $x \in X$ and $y \in Y$. Then for $a = (x,y), a' = (x',y')$ in A, we set $A(a,a') = X(x,x') \times Y(y,y')$ (where the product is taken in $\mathcal{V}$). U and V are simply the restrictions of the projections p_X and p_Y to A, both with $\mathcal{V}$-moduli given by $1_{\mathcal{V}}$.

□

Note that, for $B \in (\mathcal{V},E)$-**Cat**, a cone μ from B to the above diagram is given by two $(\mathcal{V},E)$-functors $(\mu_X, \epsilon_X) : B \to X$ and $(\mu_Y, \epsilon_Y) : B \to Y$ such that $F\mu_X$ is $\mathcal{V}$-isomorphic to $G\mu_Y$. The above construction of A as the limit of the diagram (i.e. the pullback of F and G) induces an obvious map $L_0 : B_0 \to A_0$ on the level of objects which extends to a $(\mathcal{V},E)$-functor $(L, \epsilon_B) : B \to A$ where we can take ϵ_B to be a $\mathcal{V}$-modulus that splits tensors for ϵ_X and ϵ_Y.

Proposition 9.11. *Let $X, Y, Z \in (\mathcal{V},E)$-**Cat**.*

Given the diagram

$$\begin{tikzcd} & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

if F is a saturated embedding then there is a choice of pullback (i.e. of A, U, and V

fitting into the diagram below)

$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "V"'] & X \arrow[d, "F"] \\ Y \arrow[r, "G"] & Z \end{tikzcd}$$

such that V is also a saturated embedding and FU is essentially equal to GV. Moreover, this square has the strict *universal property.*

Proof. Let the objects of A be the set $\coprod_{z \in Z} \big(F^{-1}(z) \times G^{-1}(z)\big)$. Let U and V be the restrictions to A of the projections p_X and p_Y, with $\mathcal{V}$-moduli given by $1_{\mathcal{V}}$.

□

In $(\mathcal{V}, E)$-**Cat**, if E contains $1_{\mathcal{V}} \otimes 1_{\mathcal{V}}$, then we have that $X \times Y$ is strongly $(\mathcal{V}, E)$-equivalent to $X \otimes Y$.

Remark 9.12. Note that, by essentially the same argument as in ordinary category theory, pulling back across G and then across F gives a pullback across GF (the "composite pullback"). In $(\mathcal{V}, E)$-**Cat**, if both of the individual pullbacks have the strict universal property then so does their composite.

9.1.2 Factorization in $(\mathcal{V}, E)$-Cat

We now make some observations that are relevant to the behavior of subobjects in $(\mathcal{V}, E)$-**Cat**. Again, we omit proofs when they are obvious.

Proposition 9.13. *For $X, Y \in (\mathcal{V}, E)$-**Cat** and $(F, \epsilon) : X \to Y$ a $(\mathcal{V}, E)$-functor, there is $X' \in (\mathcal{V}, E)$-**Cat** and $(F', \epsilon) : X \to X'$, $(G, 1_{\mathcal{V}}) : X' \to Y$ such that G is a saturated* *embedding and $F = GF'$ strictly (in particular up to essential equality).*

Furthermore, if F is essentially equal to $G'F''$ for some $F'' : X \to X''$ and $G' : X'' \to Y$ such that G' is a saturated embedding, then there exists a $(H, 1_{\mathcal{V}}) : X' \to X''$ such that $G = G'H$ strictly with H a saturated embedding. We call this the image *of F.*

If $F : X \to Y$ is $\mathcal{V}$-f.f. then $F' : X \to X'$ above is part of a $(\mathcal{V}, E)$-equivalence $X \underset{\tilde{F}}{\overset{F'}{\rightleftarrows}} X'$. through $\mathcal{V}$-f.f. $(\mathcal{V}, E)$-functors.

Proof. The construction is the same as in Proposition 8.11.

□

Proposition 9.14. *Let X, Y, Z be $\mathcal{V}$-categories fitting into the below diagram (of $(\mathcal{V}, E)$-functors)*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "H"'] & & Y \arrow[dl, "G"] \\ & Z & \end{tikzcd}$$

which commutes up to $\mathcal{V}$-isomorphism.

(a) *If both G and H are $\mathcal{V}$-f.f. then so is F.*

(b) *If both G and H are saturated embeddings, then there is a F', unique up to essential equality, which is $\mathcal{V}$-isomorphic to F such that F' is a saturated embedding and GF is essentially equal to H.*

(c) *If the diagram commutes up to essential equality and both G and H are saturated embeddings, then F is also a saturated embedding.*

Proposition 9.15. *Let X, Y, Z be $\mathcal{V}$-categories fitting into the below diagram (of $(\mathcal{V}, E)$-functors)*

$$X \overset{F}{\underset{G}{\rightrightarrows}} Y \xrightarrow{H} Z$$

which commutes up to $\mathcal{V}$-isomorphism, i.e. HF is $\mathcal{V}$-isomorphic to HG.

(a) If H is $\mathcal{V}$-f.f. then F is $\mathcal{V}$-isomorphic to G.

(b) If the diagram commutes up to essential equality and if H is an embedding then F is essentially equal to G.

As with the case for $\mathcal{V}$-**Cat**, Proposition 9.14 allows us, in the definition of a subobject, to consider $\iota : A \to X$ and $\iota' : A' \to X$ to represent the same object iff there is an isomorphism of $\mathcal{V}$-categories $F : A \to A'$ such that ι is essentially equal to $\iota' F$.

9.1.3 Subobjects in $(\mathcal{V}, E)$-Cat

Let $X \in (\mathcal{V}, E)$-**Cat**. We now define the category $\operatorname{Sub} X$ of subobjects of X as follows:

The objects of $\operatorname{Sub} X$ are the subobjects of X.

Given $(\mathcal{V}, E)$-functors $F : A \to B$ and $F' : A' \to B'$, we consider them equivalent when there is an isomorphism of $\mathcal{V}$-categories $G : A \to A'$ witnessing the equivalence of $\iota : A \to X$ and $\iota' : A' \to X$ as subobjects of X, along with an isomorphism of $\mathcal{V}$-categories $H : B \to B'$ witnessing the equivalence of $\theta : B \to X$ and $\theta' : B' \to X$ as subobjects of X, such that the diagram below commutes up to $\mathcal{V}$-isomorphism:

$$\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dd, "G"'] \arrow[dr, "\iota"] & & B \arrow[dd, "H"] \arrow[dl, "\theta"'] \\
& I & \\
A' \arrow[rr, "F'"'] \arrow[ur, "\iota'"] & & B' \arrow[ul, "\theta'"']
\end{tikzcd}$$

Then given $\iota : A \to X$ and $\theta : B \to X$ representing subobjects $[\iota]$ and $[\theta]$ of X, a morphism between these two subobjects is given by an equivalence class $[F]$ of a $(\mathcal{V}, E)$-functor $(F, 1_{\mathcal{V}}) : A \to B$ such that ι is $\mathcal{V}$-isomorphic to θF.

In this situation, if we hold A and B fixed then by Proposition 9.14 we have a unique representative $(F, 1_{\mathcal{V}}) : A \to B$ which is a saturated embedding such that ι is essentially equal to θF. This process is natural in A and B by uniqueness.

By the above, and by Proposition 9.15 we have that $\operatorname{Sub} X$ is in fact a poset category.

Let $(F, \epsilon) : X \to Y$. From the above observations and by Proposition 9.11 we get a functor $F^* : \operatorname{Sub} Y \to \operatorname{Sub} X$ which acts on (representatives of) subobjects of X by pullback.

For $\iota : A \to X$ representing an object of $\operatorname{Sub} X$, we can factor $F\iota$ into HG where $G : A \to \exists_F A$ is some $\mathcal{V}$-functor and $H : \exists_F A \to Y$ is the image of $F\iota$. From the properties of the image and by Proposition 9.14 it follows that this process is well-defined on $\operatorname{Sub} X$ and functorial, yielding a functor $\exists_F : \operatorname{Sub} X \to \operatorname{Sub} Y$. It is easily checked that $\exists_F$ is left adjoint to F^*.

We can construct a right adjoint $\forall_F : \operatorname{Sub} X \to \operatorname{Sub} Y$ to F^* by hand. Given $\iota : A \to X$ and the reduction map $\pi : Y \to Y_0$, take the objects of $\forall_F A$ to be the set

$$\{y \in \pi^{-1}(y_0) \mid \text{if } \pi F x = y_0, \text{ then there is some } a \in A \text{ such that } \iota a = x\}.$$

That is, it is the set of all objects of Y which are $\mathcal{V}$-isomorphic to some object $y \in Y$ satisfying the condition that every x for which Fx is $\mathcal{V}$-isomorphic to y is also "in A", i.e. of the form ιa for some $a \in A$. The inclusion into Y on the objects of $\forall_F A$ extends to a saturated embedding $\forall_F \iota : \forall_F : A \to Y$.

Now let $\theta : B \to X$ represent another subobject in $\operatorname{Sub} X$. Then $\forall_F B$ has as objects the set $\{y \in \pi^{-1}(y_0) \mid \text{if } \pi F x = y_0, \text{ then there is some } b \in B \text{ such that } \theta b = x\}$.

If there is a (representative of a) morphism in $\operatorname{Sub} X$ given by $G : A \to B$ where G is a saturated embedding and $\iota = \theta G$ strictly, then by construction we have that the set of objects of $\forall_F A$ is a subset of the set of objects of $\forall_F B$, so that the inclusion extends to a saturated embedding $\forall_F G : \forall_F A \to \forall_F B$ such that $\forall_F \iota = (\forall_F \theta) G$ strictly. This process is well-defined and functorial on $\operatorname{Sub} X$, so that we have a functor $\forall_F : \operatorname{Sub} X \to \operatorname{Sub} Y$. As with $\exists_F$, it is easy to check that $\forall_F$ right adjoint to F^*.

We record these results as a proposition:

Proposition 9.16. *For $X, Y \in (\mathcal{V}, E)$-**Cat** and a $\mathcal{V}$-functor $F : X \to Y$, the functor $F^* : \mathit{Sub}\, Y \to \mathit{Sub}\, X$ has both left and right adjoints, given by $\exists_F$ and $\forall_F$, respectively.*

9.2 Generalized subobjects

Although our current notion of subobject suffices to capture the nature of predicates in the case of **Set** (or more generally $(\mathcal{V}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ for $\mathcal{V} = \mathbb{2}$), we need to generalize the notion of subobject in order to accurately interpret predicates of continuous logic as "subobjects" in $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$. We examine the case of $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ in further detail as a guide for our intuition.

First note that all of our generalizations from $\mathcal{V}$-**Cat** to $(\mathcal{V}, E)$-**Cat** do not affect the case when $\mathcal{V} = \mathbb{2}$. That is, we get the same collection of $(\mathbb{2}, E)$-functors whether our category of $\mathcal{V}$-moduli E is the trivial one ($E_0 = \{1_{\mathbb{2}}\}$) or the maximal one ($E_m = \{1_{\mathbb{2}}, \top\}$); since also $\mathbb{2}$ is a poset, the definition of a $(\mathbb{2}, E)$-functor then coincides with that of a $\mathbb{2}$-functor.

Now subobjects of $X \in (\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ are given by saturated embeddings $\iota : A \to X$, i.e. by the commutative triangle

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\iota"'] & & X \arrow[dl, "1_X"] \\
& X &
\end{tikzcd}$$

Now recall that $1_X : X \to X$ is (represents) the terminal object of $\operatorname{Sub} X$. Since $\top$ is the initial object of the ordinary category $\mathbb{2}$ (and therefore the terminal object of $\mathbb{2}^{\mathrm{op}}$), objects of $\operatorname{Sub} X$ are equivalently objects of the (ordinary) functor category $[\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} X]$ preserving products (in particular terminal objects).

Consider the subobject classifier $* \to \Omega$ of **Set**: in $(\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ this is a $\mathbb{2}$-functor $\mathbb{I} \xrightarrow{0} \Omega$ where Ω is the symmetric $\mathbb{2}$-category with objects $\{0, 1\}$ and $\Omega(0, 1) = \perp$. That is, $\Omega = \mathbb{2}_{\mathrm{sym}} \in (\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$. Equivalently, $\mathbb{I} \xrightarrow{0} \Omega$ is the element $P \in [\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} \mathbb{2}_{\mathrm{sym}}]$ where Pa

is the saturated embedding into $\mathbb{2}_{\mathrm{sym}}$ on the objects $\{x \mid x \to a \text{ in } \mathbb{2}^{\mathrm{op}}\}$.

That Ω is a subobject classifier is expressed by the fact that P is the universal subobject, in the following sense: for any $X \in (\mathbb{2}\text{-}\mathbf{Cat})_{\mathrm{sym}}$ and $R \in \operatorname{Sub} X \simeq [\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} X]$, there is a unique $\mathbb{2}$-functor $F : X \to \mathbb{2}_{\mathrm{sym}}$ such that $R = F^*P$, where F^* is the functor $[\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} \mathbb{2}_{\mathrm{sym}}] \to [\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} X]$ induced by the pullback functor $F^* : \operatorname{Sub} \mathbb{2}_{\mathrm{sym}} \to \operatorname{Sub} X$.

In particular, for any $F : X \to \mathbb{2}_{\mathrm{sym}}$ we get a subobject $F^*P \in [\mathbb{2}^{\mathrm{op}}, \operatorname{Sub} X]$. We wish to repeat this for general $((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\mathrm{sym}}$. Henceforth when we say "$\mathcal{V}$-category" we mean "symmetric $\mathcal{V}$-category".

We will call each $R \in [\mathcal{V}^{\mathrm{op}}, \operatorname{Sub} X]$ (where $\mathcal{V}^{\mathrm{op}}$ is considered as a poset category) for which R preserves products a *$\mathcal{V}$-indexed subobject* of X (or *$\mathcal{V}$-subobject* of X, for short), and denote $[\mathcal{V}^{\mathrm{op}}, \operatorname{Sub} X]$ by $\operatorname{Sub}_{\mathcal{V}} X$. Let $P \in \operatorname{Sub}_{\mathcal{V}} \mathcal{V}_{\mathrm{sym}}$ (where $\mathcal{V}_{\mathrm{sym}}$ is considered as a $\mathcal{V}$-category) be given on the objects of $\mathcal{V}^{\mathrm{op}}$ as follows: for $a \in \mathcal{V}^{\mathrm{op}}$, let Pa be the saturated embedding into $\mathcal{V}_{\mathrm{sym}}$ on the objects $\{x \in \mathcal{V}_{\mathrm{sym}} \mid x \to a \text{ in } \mathcal{V}^{\mathrm{op}}\}$. For any $(\mathcal{V}, E)$-functor $F : X \to \mathcal{V}_{\mathrm{sym}}$ we get an induced $F^*P \in \operatorname{Sub}_{\mathcal{V}} X$. More generally, for a $(\mathcal{V}, E)$-functor $F : X \to Y$ we have $F^* : \operatorname{Sub}_{\mathcal{V}} Y \to \operatorname{Sub}_{\mathcal{V}} X$. It is easy to see that the left (resp. right) adjoint $\exists_F$ (resp. $\forall_F$) $: \operatorname{Sub} X \to \operatorname{Sub} Y$ extends (through pointwise application) to a left (resp. right) adjoint $\exists_F$ (resp. $\forall_F$) $: \operatorname{Sub}_{\mathcal{V}} X \to \operatorname{Sub}_{\mathcal{V}} Y$ to F^*.

Definition 9.17. Given X a $\mathcal{V}$-category and $R \in \operatorname{Sub}_{\mathcal{V}} X$, define $\bar{R} \in \operatorname{Sub}_{\mathcal{V}} X$ as follows:

For each $a \in \mathcal{V}^{\mathrm{op}}$, set $\bar{R}a$ to be the saturated embedding into X on the objects $\{x \in X \mid \coprod_{y \in Ra} X(x, y) = I\}$ (where the coproduct is taken in $\mathcal{V}$). This is natural in a, giving $\bar{R} \in \operatorname{Sub}_{\mathcal{V}} X$. We call $\bar{R}$ the *closure* of R in X.

By construction, we have that $R \to \bar{R}$ in $\operatorname{Sub}_{\mathcal{V}} X$; if $R = \bar{R}$ then we say that R is *closed*. Note that $\bar{\bar{R}} = \bar{R}$, so that $\bar{R}$ is closed. Furthermore we have that if $R \to R'$ in $\operatorname{Sub}_{\mathcal{V}} X$ then

$\bar{R} \to \overline{R'}$ so that $R \mapsto \bar{R}$ extends to a functor $\overline{(-)} : \mathrm{Sub}_{\mathcal{V}} X \to \mathrm{Sub}_{\mathcal{V}} X$, which is in fact an idempotent monad on $\mathrm{Sub}_{\mathcal{V}} X$. Equivalently, the full subcategory of $\mathrm{Sub}_{\mathcal{V}} X$ on the closed $\mathcal{V}$-subobjects is reflective.

For convenience, we may also say that some individual subobject $R \in \mathrm{Sub}\, X$ may be closed if the obvious constant-on-$\mathcal{V}^{\mathrm{op}}$ $\mathcal{V}$-subobject $R \in \mathrm{Sub}_{\mathcal{V}} X$ that it defines is closed.

Proposition 9.18. *For a $(\mathcal{V}, E)$-functor $(F, \epsilon) : X \to Y$ and $R \in Sub_{\mathcal{V}} Y$, we have that $\overline{F^* R} \to F^* \bar{R}$ in $Sub_{\mathcal{V}} X$.*

Proof. Since $(F^*R)a = F^*(Ra)$, it suffices to check that $\overline{F^*Ra} \to F^*(\bar{R}a)$ in $\mathrm{Sub}\, X$ for given $a \in \mathcal{V}^{\mathrm{op}}$. By the nature of $\mathrm{Sub}\, X$ it suffices to check this on the sets of objects, which by abuse of notation we also denote by $\overline{F^*Ra}$ and $F^*(\bar{R}a)$.

$F^*(\bar{R}a)$ is given by the set $\{x \mid Fx \in \bar{R}a\} = \{x \mid \coprod_{y \in Ra} Y(Fx, y) = I\}$.

$\overline{F^*Ra}$ is given by the set $\{x \mid \coprod_{x' \in F^*Ra} X(x, x') = I\}$.

Let $x \in \overline{F^*Ra}$. Then $\coprod_{x' \in F^*Ra} X(x, x') = I$. Since by assumption ϵ preserves I as a colimit, we have that $\coprod_{x' \in F^*Ra} \epsilon(X(x, x')) = I$.

We have $\epsilon(X(x, x')) \to Y(Fx, Fx')$ for each x, x' and $x' \in F^*Ra$ iff $Fx' \in Ra$, so that we have $\coprod_{x' \in F^*Ra} \epsilon(X(x, x')) \to \coprod_{x' \in F^*Ra} Y(Fx, Fx') \to \coprod_{y \in Ra} Y(Fx, y)$. Since $\mathcal{V}$ is a poset and I is the terminal object, this implies $\coprod_{y \in Ra} Y(Fx, y) = I$. Thus $x \in F^*(\bar{R}a)$.

□

The above implies the analogue of the familiar result from basic analysis that if R is closed, then so is F^*R.

If $P \in \mathrm{Sub}_{\mathcal{V}} \mathcal{V}_{\mathrm{sym}}$ is closed (which it certainly is in our cases of interest), each $(\mathcal{V}, E)$-functor $X \to \mathcal{V}_{\mathrm{sym}}$ then determines a closed $\mathcal{V}$-subobject $F^*P \in \mathrm{Sub}_{\mathcal{V}} X$. We want to determine the extent to which each closed $\mathcal{V}$-subobject of X determines a $(\mathcal{V}, E)$-functor $X \to \mathcal{V}_{\mathrm{sym}}$.

The answer to our question will depend on our choice of $\mathcal{V}$ and E; to make the problem tractable, we specialize to our case of interest, namely $\mathcal{V} = \mathbb{R}$ and $E = E_m$. Then every $\epsilon \in E_m$ is a function $[0, \infty] \to [0, \infty]$ that fixes 0 and is continuous at 0, so that $(\mathbb{R}, E_m)$-functors are exactly uniformly continuous functions. In this case, $A \in \operatorname{Sub} X$ being closed is exactly the same as A being closed as a subspace of the metric space X. For each $a \in \mathbb{R}$, $Pa = [0, a] \hookrightarrow [0, \infty] = \mathbb{R}$. (We will henceforth refer to $(\mathbb{R}, E_m)$-functors $F : X \to Y$ interchangably as uniformly continuous functions $X \to Y$.) Then if R is a $\mathbb{R}$-subobject of X that can be obtained as F^*P for some uniformly continuous $F : X \to \mathbb{R}_{\mathrm{sym}}$ then for each $a \in \mathbb{R}^{\mathrm{op}}$ the set of objects of Ra is of the form $F^{-1}([0, a])$. Clearly not all $R \in \operatorname{Sub}_{\mathbb{R}} X$ can be obtained as F^*P for a uniformly continuous $F : X \to \mathbb{R}_{\mathrm{sym}}$, since we may imagine that some $R \in \operatorname{Sub}_{\mathbb{R}} X$ may give the "sublevel sets" of a discontinuous function $f : X \to [0, \infty]$.

9.2.1 Compactness in $(\mathbb{R}, E_m)$-Cat

As is done for the metric spaces considered in continuous logic, we will require that the $\mathbb{R}$-categories in our domain of discourse satisfy a compactness property. First note that for our metric spaces, which are actually extended pseudometric spaces, there are two competing notions of compactness, *metric* and *topological*. X as an extended pseudometric space is metrically compact when it has the property that for any sequence $\{x_n\}$ of points of X, there is some $x \in X$ such that $\lim\limits_{n \to \infty} d(x, x_n) = 0$. This is certainly not equivalent to topological compactness, which is the usual notion applied to X viewed as a topological space; a sequence that converges metrically converges topologically, while the converse fails in general. These two notions of course coincide when X is an ordinary (non-extended) metric space, but for our purposes we will need metric compactness, which is equivalent to being Cauchy complete and totally bounded.

Now from [7] we know that for any $\mathbb{R}$-category X, $\mathbb{R}$-$\mathbf{Cat}(X^{\mathrm{op}}, \mathbb{R})$ is Cauchy complete as a metric space. Now for any E, $\mathbb{R}$-$\mathbf{Cat}(X^{\mathrm{op}}, \mathbb{R})$ is just the quotient of $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$ by the relation of being essentially equal, which yields a $(\mathbb{R}, E)$-equivalent $\mathbb{R}$-category. Thus $\mathbb{R}$-$\mathbf{Cat}(X^{\mathrm{op}}, \mathbb{R})$ being Cauchy complete implies that $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$ is also.

For any $X \in (\mathbb{R}, E_m)\text{-}\mathbf{Cat}$, we know (say from [27]) that it embeds isometrically into $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$ via the Yoneda embedding. In more detail,

$$(\mathcal{Y}, 1_\mathbb{R}) : X \to ((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$$

given by $a \mapsto (X(-, a), 1_\mathbb{R})$ is $\mathcal{R}$-f.f. By Proposition 9.13 we can replace this by a (unique up to strong equivalence fixing the codomain) saturated embedding

$$\mathcal{Y}' : X' \to ((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R}),$$

with X equivalent to X' through $\mathbb{R}$-f.f. functors. Then X is Cauchy complete as a metric space iff X' is, and Cauchy completeness of X' is equivalent to X' being closed in $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$.

Henceforth we require all $\mathbb{R}$-categories X in our domain of discourse to satisfy the condition that X' (with notation as above) be closed as a subobject of $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{1_\mathbb{R}}(X^{\mathrm{op}}, \mathbb{R})$. In particular, this implies that for each X, each $A \in \operatorname{Sub} X$ that is closed is also Cauchy complete as a metric space.

It is possible for us to state the condition of being totally bounded in categorical language, but for our purposes it is a condition most naturally stated in the analytic language of metric spaces. This is unproblematic since the objects of our study - objects of the category $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$ - are actual metric spaces. Thus we require that all $X \in ((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$ in our discourse to satisfy the condition that every sequence in X has a Cauchy subsequence.

The above conditions are equivalent to requiring that all $\mathbb{R}$-categories X are metrically compact when viewed as metric spaces; we will therefore call them *compact* as objects of $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$. Thus all the usual results from analysis that apply to compact (pseudo)metric spaces apply to the $\mathbb{R}$-categories that we will consider (such as the property of compactness being closed under finite products).

In particular, we consider the following: for any $X \in ((\mathcal{V}, E)\text{-}\mathbf{Cat})_{\text{sym}}$ we have the $(\mathcal{V}, E)$-functor $d = X(-,-) : X \otimes X \to \mathcal{V}_{\text{sym}}$ given on objects by $d(a_1, a_2) = X(a_1, a_2)$. If E contains $1_{\mathcal{V}} \otimes 1_{\mathcal{V}}$ then pulling back across the strong equivalence $X \times X \to X \otimes X$ gives us $d : X \times X \to \mathcal{V}_{\text{sym}}$. In the case that $\mathcal{V} = \mathbb{R}$ and $E = E_m$ this is just the distance function on X.

Since all $\mathbb{R}$-categories X in our consideration are metrically compact, they are certainly bounded as metric spaces, and thus for each such X there is some $a \neq \infty$ in $\mathbb{R}^{\text{op}}$ such that $d^* Pa$ is the terminal object (i.e. the whole space $X \times X$) of $\operatorname{Sub}(X \times X)$ (which is equivalent to having $\exists_d (X \times X) \to Pa$ in $\operatorname{Sub} \mathbb{R}_{\text{sym}}$).

9.3 The main results

As one consequence of our assumptions, every $(F, \epsilon) : X \to \mathbb{R}_{\text{sym}}$ factors as $i \circ (F, \epsilon, B)$ for $(F, \epsilon, B) : X \to PB = [0, B]$ and $i : [0, B] \hookrightarrow \mathbb{R}_{\text{sym}}$, for some $B \neq \infty$. Thus any $(F, \epsilon) : X \to \mathbb{R}_{\text{sym}}$ may equivalently be considered as $(F, \epsilon) : X \to [0, B]$. We will consider the datum B to be part of background bookkeeping; when relevant we will mention it explicitly.

Proposition 9.19. *Let A, X be $\mathbb{R}$-categories. For any uniformly continuous $F : A \times X \to \mathbb{R}_{sym}$, we have the following:*

(a) For each $a \in \mathbb{R}^{op}$, the set of objects of $\exists_{\pi_X} F^* Pa \in Sub\, X$ is equal to

$$\{x \in X \mid \inf_{y \in A} F(y, x) \leq a\}$$

(b) For each $a \in \mathbb{R}^{op}$, the set of objects of $\forall_{\pi_X} F^* Pa \in Sub\, X$ is equal to

$$\{x \in X \mid \sup_{y \in A} F(y, x) \leq a\}$$

Proof. With notation as above, our construction of $\exists_{\pi_X}$, F^*, and P gives us that the set of objects of $\exists_{\pi_X} F^* Pa \in \mathrm{Sub}\, X$ is given by $\{x \in X \mid \exists y \in A \text{ such that } F(y, x) \leq a\}$. By compactness of A we have that this last set is equal to $\{x \in X \mid \inf_{y \in A} F(y, x) \leq a\}$.

The proof for $\forall_{\pi_X} F^* Pa$ is obvious.

$\square$

Remark 9.20. The above shows that subobjects of the form $\exists_{\pi_X} F^* Pa$ and $\forall_{\pi_X} F^* Pa$ are closed.

Given $A \in \mathrm{Sub}_{\mathbb{R}} X$, each $a \in \mathbb{R}^{\mathrm{op}}$ determines an element $D^A(a, -) \in \mathrm{Sub}_{\mathbb{R}} X$ as follows. Denote by $i_a : Aa \times X \to X \times X$ the obvious inclusion. Then $(d \circ i_a)^* P \in \mathrm{Sub}_{\mathbb{R}} (A_a \times X)$. Letting $\pi_X : A_a \times X \to X$ denote the projection onto X, define

$$D^A(a, -) = \exists_{\pi_X} (d \circ i_a)^* P \in \mathrm{Sub}_{\mathbb{R}} X.$$

For each $\delta \in \mathbb{R}^{\mathrm{op}}$, $D^A(a, \delta) \in \mathrm{Sub}\, X$ is the subspace of X consisting of the points $x \in X$ for which $\inf_{y \in A_a} X(y, x) \leq \delta$. This is natural in a, so that $D^A \in [\mathbb{R}^{\mathrm{op}} \times \mathbb{R}^{\mathrm{op}}, \mathrm{Sub}\, X]$. For each $\epsilon \in E_m$ there is another element $A_\epsilon \in [\mathbb{R}^{\mathrm{op}} \times \mathbb{R}^{\mathrm{op}}, \mathrm{Sub}\, X]$ determined by A, given by $(a, \delta) \mapsto A_{a+\epsilon(\delta)}$.

9.3.1 A subobject classifier

We are now ready to give the main statement of the correspondence between $\mathrm{Sub}_{\mathbb{R}} X$ and functors $(F, \epsilon) : X \to \mathbb{R}_{\mathrm{sym}}$, which exhibits $P \in \mathrm{Sub}_{\mathbb{R}} \mathbb{R}_{\mathrm{sym}}$ as a kind of subobject classifier, analogous to the way that $\top \to \Omega$ is the subobject classifier for **Set**.

Theorem 9.21. *Let X be a $\mathbb{R}$-category.*

Given $(F, \epsilon) : X \to \mathbb{R}_{sym}$, we have that $D^{F^*P} \to (F^*P)_\epsilon$ in $[\mathbb{R}^{op} \times \mathbb{R}^{op}, Sub\, X]$.

Conversely, for any $A \in Sub_{\mathbb{R}} X$, if there is some $\epsilon \in E_m$ such that $D^A \to A_\epsilon$ in $[\mathbb{R}^{op} \times \mathbb{R}^{op}, Sub\, X]$, there is some $(F, \epsilon) : X \to \mathbb{R}_{sym}$ (necessarily unique up to essential equality) for which $A = F^*P$. We denote this F by A_*.

Proof. Let $(F, \epsilon) : X \to \mathbb{R}_{\mathrm{sym}}$ be given. For each $(a, \delta) \in \mathbb{R}^{\mathrm{op}} \times \mathbb{R}^{\mathrm{op}}$,

$$D^{F^*P}(a, \delta) = \{x \in X \mid \inf_{y \in F^*Pa} X(x, F^*Pa) \leq \delta\}.$$

F^*Pa is closed in X so it is compact, and so there is some $x' \in F^*Pa$ for which $X(x, x') \leq \delta$. Now $Fx' \leq a$ in $\mathbb{R}_{\text{sym}}$ and must have that $\mathbb{R}_{\text{sym}}(Fx, Fx') \leq \epsilon(\delta)$, so $Fx \leq a + \epsilon(\delta)$. Thus $x \in (F^*P)_\epsilon(a, \delta)$, and we must have $D^{F^*P} \to (F^*P)_\epsilon$.

Now let $A \in \operatorname{Sub}_{\mathbb{R}} X$ be given, along with some $\epsilon \in E_m$ such that $D^A \to A_\epsilon$ in $[\mathbb{R}^{\text{op}} \times \mathbb{R}^{\text{op}}, \operatorname{Sub} X]$. Define, for each $x \in X$, $Fx = \inf\{a \in \mathbb{R}^{\text{op}} \mid x \in Aa\}$. We must exhibit ϵ as a $\mathbb{R}$-modulus for F, i.e. we must verify for each $x, x' \in X$ that $\epsilon(X(x, x')) \to \mathbb{R}_{\text{sym}}(Fx, Fx')$ in $\mathbb{R}$. Assume w.l.o.g. that $Fx \leq Fx'$ in $\mathbb{R}_{\text{sym}}$. Then $\mathbb{R}_{\text{sym}}(Fx, Fx') = Fx' - Fx = \inf\{a \in \mathbb{R}^{\text{op}} \mid x' \in Aa\} - \inf\{a \in \mathbb{R}^{\text{op}} \mid x \in Aa\}$.

Let $X(x, x') = \delta$. Now for every $c > 0$, there is some a_c such that $x \in A_{a_c}$ and $\inf\{a \in \mathbb{R}^{\text{op}} \mid x' \in Aa\} - \inf\{a \in \mathbb{R}^{\text{op}} \mid x \in Aa\} < \inf\{a \in \mathbb{R}^{\text{op}} \mid x' \in Aa\} - a_c + c$. Since $x \in A_{a_c}$ implies $x' \in A_{a_c + \epsilon(\delta)}$, we have that $\inf\{a \in \mathbb{R}^{\text{op}} \mid x' \in Aa\} - a_c + c \leq \epsilon(\delta) + c$. Letting $c \to 0$, we get $\mathbb{R}_{\text{sym}}(Fx, Fx') \leq \epsilon(\delta)$, i.e. $\epsilon(X(x, x')) \to \mathbb{R}_{\text{sym}}(Fx, Fx')$ in $\mathbb{R}$.

Fix $a \in \mathbb{R}^{\text{op}}$. F^*Pa as given above is $\{x \in X \mid \inf\{a' \in \mathbb{R}^{\text{op}} \mid x \in Aa'\} \leq a\}$. Clearly $Aa \to F^*Pa$ in $\operatorname{Sub} X$. Now for any $c > 0$ we have that $F^*Pa \to A_{a+c}$ in $\operatorname{Sub} X$. By assumption we have that $\lim\limits_{c \to 0} A_{a+c} = A_a$ in $\operatorname{Sub} X$, so that $F^*Pa \to A_a$ in $\operatorname{Sub} X$, so that $F^*Pa = A_a$.

$\square$

Given $A \in \operatorname{Sub}_{\mathbb{R}} X$ satisfying the hypothesis of the second part of the above theorem, we can require that the ϵ guaranteed by the theorem be part of the data specifying $A \in \operatorname{Sub}_{\mathbb{R}} X$ (its "$\mathbb{R}$-modulus"), so that we write $(A, \epsilon) \in \operatorname{Sub}_{\mathbb{R}} X$. We call (A, ϵ) a *continuous* $\mathbb{R}$-subobject of X; thus when we write (A, ϵ) it is implied that it is a continuous $\mathbb{R}$-subobject with $\mathbb{R}$-modulus ϵ.

For any such $(A, \epsilon) \in \operatorname{Sub}_{\mathbb{R}} X$ we have that A_* (which has $\mathbb{R}$-modulus given by the same

ϵ) is not only unique up to essential equality, but $(A_*)^*\exists_{A_*}A = A$ since $A = (A_*)^*P$ and $\exists_{A_*}, (A_*)^*$ give a (covariant) Galois connection.

Now by compactness of X we have that the image of A_* in $\mathbb{R}_{\text{sym}}$ factors through $[0, a]$ for some $a \neq \infty$. This is equivalent to $A \in \operatorname{Sub}_{\mathbb{R}} X$ satisfying $Aa' = X$ for every $a \to a'$ in $\mathbb{R}^{\text{op}}$ (we call this a the (a) *predicate bound* of A). We may also require that this be part of the data specifying A, so that we write (A, ϵ, a) to specify a continuous $\mathbb{R}$-subobject of X, with predicate bound a.

We give one final construction before giving the interpretation of continuous logic into $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$:

Definition 9.22. Let $a_1, a_2 \in [0, \infty)$.

$(\dot{-}, 1_{\mathbb{R}}) : [0, a_1] \times [0, a_2] \to \mathbb{R}_{\text{sym}}$ is defined by $(x, y) \mapsto \max(x - y, 0)$. We write this as $x \dot{-} y$. Clearly the image of $\dot{-}$ restricts to $[0, a_1]$, so that we may consider it as a function $(\dot{-}, 1_{\mathbb{R}}) : [0, a_1] \times [0, a_2] \to [0, a_1]$.

Let $X, Y \in ((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$. Given $R \in \operatorname{Sub}_{\mathbb{R}} X$ and $R' \in \operatorname{Sub}_{\mathbb{R}} Y$ with corresponding $R_* : X \to \mathbb{R}_{\text{sym}}$ and $R'_* : Y \to \mathbb{R}_{\text{sym}}$, we can form $R_* \dot{-} R'_* = \dot{-} \circ (R_* \times R'_*) : X \times Y \to \mathbb{R}_{\text{sym}}$.

9.4 Interpreting continuous logic into $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\text{sym}}$

9.4.1 Syntax of continuous logic

Recall from Part I the basic syntax of single-sorted continuous logic. This generalizes straightforwardly to the many-sorted case, which we describe now explicitly.

Definition 9.23. A *continuous signature* S consists of:

(a) A set $\mathcal{S}$ of *sort symbols* s_i (containing $*$, the terminal sort), each with:

(i) A corresponding *metric symbol* d_i;

(ii) A corresponding *bound* B_i, a nonnegative finite real number;

(b) A set $\mathcal{F}$ of *function symbols* f_j, such that for each $f \in \mathcal{F}$ we have the following data:

(i) A natural number n and an $(n+1)$-tuple $(s_1, \ldots, s_n, s)$ of elements of $\mathcal{S}$;

(ii) A *modulus of uniform continuity* ϵ, i.e. a monotonic function $\epsilon : [0, \infty] \to [0, \infty]$ which is continuous at 0 and has $\epsilon(0) = 0$; with the above we say that (f, ϵ) is an n-ary function symbol of type $\left(\prod_{1 \leq i \leq n} s_i\right) \to s$, or write $(f, \epsilon) : \left(\prod_{1 \leq i \leq n} s_i\right) \to s$

(c) A set $\mathcal{R}$ of *predicate symbols* R_k, such that for each $R \in \mathcal{R}$ we have the following data:

(i) A positive natural number n and an n-tuple $(s_1, \ldots, s_n)$ of elements of $\mathcal{S}$;

(ii) A *modulus of uniform continuity* ϵ, i.e. a monotonic function $\epsilon : [0, \infty] \to [0, \infty]$ which is continuous at 0 and has $\epsilon(0) = 0$;

(iii) A *predicate bound*, i.e. a nonnegative (finite) real number a; with the above we say that (R, ϵ, a) is an n-ary predicate symbol of type $\prod_{1 \leq i \leq n} s_i$, or write $(R, \epsilon, a) \subset_{\mathbb{R}} \prod_{1 \leq i \leq n} s_i$.

In addition to the symbols provided by our signature S, we have, for each natural number n and an $(n+1)$-tuple $(a_1, \ldots, a_n, a)$ of nonnegative finite real numbers, a symbol and corresponding modulus of uniform continuity for each continuous function $(u_a^{(a_i)}, \epsilon) : \left(\prod_{1 \leq i \leq n} [0, a_i]\right) \to [0, a]$, which we call a *connective*. Definition 9.22 therefore defines a binary connective.

As previously, the empty product $\prod_{\emptyset} s_i$ is understood to be $*$. A 0-ary function symbol $c : * \to s$ is also called a *constant symbol of type s*.

A 0-ary connective r is simply a constant nonnegative (finite) real number r, which we call just that.

Furthermore, for each $s \in \mathcal{S}$ we require an infinite set $\{x_i\}$ of *variables of type s*.

The construction of terms and formulas for a signature S is done analogously to single-sorted continuous logic:

Definition 9.24. Let S be a continuous signature.

(a) A *term* for S is given by the following inductive description:

  (i) Each variable x of type s is a term of type s and modulus $1_{\mathbb{R}}$, with free variable x.

  (ii) Each constant symbol c of type s is a term of type s, with no free variables.

  (iii) Let $t_1, \ldots, t_n$ be terms where t_k is of type s_k and modulus ϵ_k, and $(f, \epsilon) : \prod_{1 \le k \le n} s_k \to s$.
  Then $f(t_1, \ldots, t_n)$ is a term of type s, with free variables given by the union over k of the free variables of each t_k, and modulus $\epsilon \circ (\max_k(\epsilon_k))$.

(b) Let $t_1, \ldots, t_n$ be terms where t_k is of type s_k, and $(R, \epsilon, a) \subset_{\mathbb{R}} \prod_{1 \le k \le n} s_k$.

Then $R(t_1, \ldots, t_n)$ is an *atomic formula* with free variables given by the union over k of the free variables of each t_k, and has modulus $\epsilon \circ (\max_k(\epsilon_k))$ and predicate bound a. (For each $s \in \mathcal{S}$, the distance symbol d_s is treated as a binary predicate symbol of type $s \times s$, with modulus $1_{\mathbb{R}}$ and predicate bound B_s.)

(c) A *formula* for S is given by the following inductive description:

  (i) Each atomic formula is a formula.

(ii) If $\phi_1, \ldots, \phi_2$ are formulas with respective moduli $\epsilon_1, \ldots, \epsilon_n$, and predicate bounds $a_1, \ldots, a_n$; and if $u : \prod_{1 \leq i \leq n} [0, a_i] \to [0, a]$ is a connective with modulus ϵ, then $u(\phi_1, \ldots, \phi_n)$ is a formula with modulus $\epsilon \circ (\max_i(\epsilon_i))$ and predicate bound a. Its free variables are given by the union of the free variables of $\phi_1, \ldots, \phi_n$.

If $n = 2$ and u is the connective $\dot{-}$ as defined in Definition 9.22, then we write $u(\phi_1, \phi_2)$ as $\phi_1 \dot{-} \phi_2$ or as $\phi_1 \leq \phi_2$.

(iii) If ϕ is a formula with modulus ϵ and predicate bound a, and x is a free variable of ϕ, then $\sup_x \phi$ and $\inf_x \phi$ are formulas, each with modulus ϵ and predicate bound a, with free variables equal to the free variables of ϕ omitting x.

(iv) A formula with no free variables is called a *sentence*.

(d) A *condition* for S is a formula of the form $\phi = 0$ (which may be given as $\phi \leq 0$).

(e) A condition with no free variables is called *closed*.

An S-theory Σ is a set of closed S-conditions.

9.4.2 The interpretation

Given a continuous signature S, we give its interpretation into $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$, which will formally resemble the interpretation of classical logic into **Set**; we have designed our framework with this intention. Recall that by $P \in \mathrm{Sub}_{\mathbb{R}}\, \mathbb{R}_{\mathrm{sym}}$ we mean the distinguished $\mathbb{R}$-subobject of $\mathbb{R}_{\mathrm{sym}}$ given by $Pa = [0, a]$. We will abuse notation by using P to also denote $i^*P \in \mathrm{Sub}_{\mathbb{R}}\, Pa$ where $i : Pa \to \mathbb{R}_{\mathrm{sym}}$ is the inclusion.

Definition 9.25. Given a continuous signature S, an S-*structure* is given by the following:

(a) For each sort symbol $s \in \mathcal{S}$, a compact object $[\![s]\!] \in ((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$, such that $[\![*]\!] = \mathbb{I}$.

(a) We require that the distance function $d_{[\![s]\!]} = [\![s]\!](-,-) : [\![s]\!] \times [\![s]\!] \to \mathbb{R}_{\mathrm{sym}}$ factors through $d_{[\![s]\!]} : [\![s]\!] \times [\![s]\!] \to [0, B_s]$ (i.e. that $[\![s]\!]$ is bounded in diameter by B_s).

(b) For each function symbol $(f, \epsilon) : \prod_i s_i \to s$, an $(\mathbb{R}, E)$-functor $([\![f]\!], \epsilon) : \prod_i [\![s_i]\!] \to [\![s]\!]$.

(c) For each predicate symbol $(R, \epsilon, a) \subset_{\mathbb{R}} \prod_i s_i$, a $\mathbb{R}$-subobject $([\![R]\!], \epsilon, a) \in \mathrm{Sub}_{\mathbb{R}}\, X$.

For $s \in \mathcal{S}$, its corresponding metric symbol d_s is treated as a binary predicate symbol $(d_s, 1_{\mathbb{R}}, B_s)$ where B_s is the corresponding bound for S.

Where in the above we have implicitly made a choice of products. The data then determine the interpretation of all S-terms and S-formulas, as follows. As before, any time we have a tuple $\vec{x} = (x_1, \ldots, x_n)$ of distinct variables of types $s_1, \ldots, s_n$ (respectively), we set $[\![\vec{x}]\!] = \prod_{1 \le i \le n} [\![s_i]\!]$. In particular, if a variable x is of type s then $[\![x]\!]$ is $[\![s]\!]$, and if $\vec{x}$ is empty then $[\![x]\!] = [\![*]\!]$.

If $t = x_i$ then $([\![t]\!]_{\vec{x}}, 1_{\mathbb{R}}) : [\![\vec{x}]\!] \to [\![x_i]\!]$ is the projection map.

Let f be some n-ary function symbol with modulus ϵ. If $t = f(t_1, \ldots, t_n)$ with each t_i of type s_i, with each $[\![t_i]\!]_{\vec{x}}$ already defined with modulus ϵ_i, then $([\![t]\!]_{\vec{x}}, \epsilon \circ \max_i(\epsilon_i)) : [\![\vec{x}]\!] \to [\![s]\!]$ is given by the composition $[\![f]\!] \circ ([\![t_1]\!]_{\vec{x}}, \ldots, [\![t_n]\!]_{\vec{x}})$.

We interpret each formula ϕ with free variables among $\vec{x}$ as a $\mathbb{R}$-subobject $[\![\phi]\!]_{\vec{x}} \in \mathrm{Sub}_{\mathbb{R}}\, [\![\vec{x}]\!]$, given by the following:

If ϕ is the atomic formula $d_s(t_1, t_2)$ with both t_1 and t_2 terms of type s and respective moduli ϵ_1 and ϵ_2, then $([\![\phi]\!]_{\vec{x}}, \max(\epsilon_1, \epsilon_2), B_s) \in \mathrm{Sub}_{\mathbb{R}}\, [\![\vec{x}]\!]$ is given by $\left(d_{[\![s]\!]} \circ ([\![t_1]\!], [\![t_2]\!])\right)^* P$.

Let R be some n-ary predicate symbol with modulus ϵ and predicate bound a. If ϕ is an atomic formula $R(t_1, \dots, t_n)$ where each t_i has sort s_i and modulus ϵ_i, then $(\llbracket\phi\rrbracket_{\vec{x}}, \epsilon\circ(\max_i \epsilon_i), a) \in \mathrm{Sub}_{\mathbb{R}} \llbracket\vec{x}\rrbracket$ is given by $(\llbracket R\rrbracket_* \circ (\llbracket t_1\rrbracket, \dots, \llbracket t_n\rrbracket))^* P = (\llbracket t_1\rrbracket, \dots, \llbracket t_n\rrbracket)^* \llbracket R\rrbracket$.

Given interpretations $(\llbracket\phi_i\rrbracket_{\vec{x}}, \epsilon_i, a_i)_{\vec{x}}$ for $1 \le i \le n$ and a n-ary connective $\left(u_a^{(a_i)}, \epsilon\right)$, we set $\left(\llbracket u(\phi_1, \dots, \phi_n)\rrbracket, \epsilon \circ \max_i(\epsilon_i), a\right) \in \mathrm{Sub}_{\mathbb{R}} \llbracket\vec{x}\rrbracket$ as $(u \circ (\llbracket\phi_1\rrbracket_*, \dots, \llbracket\phi_n\rrbracket_*))^* P$.

Given an interpretation $(\llbracket\phi\rrbracket_{\vec{x},y}, \epsilon, a)$ of ϕ, we set $(\llbracket\inf_y \phi\rrbracket_{\vec{x}}, \epsilon, a) \in \mathrm{Sub}_{\mathbb{R}} \llbracket\vec{x}\rrbracket$ as $\exists_\pi \llbracket\phi\rrbracket_{\vec{x}}$ and $(\llbracket\sup_y \phi\rrbracket_{\vec{x}}, \epsilon, a) \in \mathrm{Sub}_{\mathbb{R}} \llbracket\vec{x}\rrbracket$ as $\forall_\pi \llbracket\phi\rrbracket_{\vec{x}}$, where $\pi : \llbracket\vec{x}\rrbracket \times \llbracket y\rrbracket \to \llbracket\vec{x}\rrbracket$ is the obvious projection.

Remark 9.26. It is straightforward to check, for example using Proposition 9.19 and the proof of Theorem 9.21, that $\llbracket\inf_y \phi\rrbracket = (\inf_y \phi(-, y))^* P$, where $\phi(-, -) = \llbracket\phi\rrbracket_*$.

This completes the interpretation of all S-terms and S-formulas given an interpretation of the signature S into $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$. If ϕ is a sentence, then $\llbracket\phi\rrbracket_\emptyset \in \mathrm{Sub}_{\mathbb{R}} \llbracket *\rrbracket$ is an $\mathbb{R}$-subobject of $\mathbb{I}$. If ϕ is a closed condition then we say that ϕ is *true* in this interpretation if $\llbracket\phi\rrbracket_\emptyset$ is the terminal object in $\mathrm{Sub}_{\mathbb{R}} \llbracket *\rrbracket$.

If Σ is a collection of S-conditions, we say that an interpretation of S is a *model* of Σ if the interpretation makes each $\phi \in \Sigma$ true in that interpretation.

Chapter 10

Future directions

10.1 Ultraproducts

Continuous logic, as with classical logic, makes extensive use of ultraproducts in both theory and practice. There already exists a notion of taking ultraproducts in topoi that corresponds to the classical ultraproduct (the "filter-quotient construction" [31]); extending such a construction to this continuous setting that corresponds to the continuous ultraproduct would be an interesting and potentially fruitful endeavor.

10.2 $\mathcal{W}(\mathcal{C})$-enrichment

The notion of $(\mathcal{V}, E)$-**Cat** being $((\mathcal{V}, E)\text{-}\mathbf{Cat})(\mathcal{C})$-enriched naturally extends to the question of what it means for $(\mathcal{V}, E)$-**Cat** to be "closed" with respect to this enrichment. In particular, the ordinary category $(\mathbf{pMet}_\infty)_u$ (which is morally the same as $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$) fails to be cartesian closed, but it may be the case that $(\mathbb{R}, E_m)$-**Cat** (or $((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}}$) is closed with respect to its "$\mathcal{C}$-indexed" enrichment.

10.3 Generalize to $((\mathbb{R}, E_m)$-$\mathbf{Cat})_{\mathbf{sym}}(\mathcal{C})$-categories

In a different direction, just as the logic corresponding to that of **Set**, which is enriched over itself, may be interpreted in other categories (which are still enriched over **Set**) that bear specific structural similarities to **Set**, we may ask to what extent continuous logic may be interpreted into general $(((\mathbb{R}, E_m)\text{-}\mathbf{Cat})_{\mathrm{sym}})(\mathcal{C})$-categories, and what kinds of conditions such categories would need to satisfy.

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