Hyperbolic Geometry and Algebraic geometry, Seoul-Austin, 2014/15

Igor Dolgachev

December 4, 2019

Contents

Introduction

This is an extending version of my lectures in Seoul in October 2014 and in Austin in February 2015. The main topic covered in the lectures is an interrelationship between the theory of discrete groups acting in hyperbolic spaces and groups of automorphisms of algebraic varieties. Also I explain in detail some application of hyperbolic geometry to the problem of counting the degrees of curves in an orbit of an infinite group of automorphisms of an algebraic surfaces. The background in algebraic geometry for experts in hyperbolic geometry and vice versa was assumed to be minimal. I refer to many results in hyperbolic/algebraic geometry without references. The main sources for these references are [1] and [30].

It is my pleasure to thank the audience for their patience and motivating questions. I also thank JongHae Keum and Daniel Allcock for inviting me to give the lectures.

Lecture 1

Hodge-Index Theorem

A *Minkowski* (or *Lorentzian*, or *pseudo-Euclidean*) is a real vector space V of finite dimension $n+1>1$ equipped with a nondegenerate symmetric bilinear form (v,w) of signature $(1,n)$ (or $(n,1)$) but we stick to $(1,n)$). The signature of this type is called *hyperbolic*. One can choose a basis $(e_1,\ldots,e_{n+1})$ in V such that the matrix of the bilinear forms becomes the diagonal matrix $\mathrm{diag}(1,-1,\ldots,-1)$.

An *integral structure* on V is defined by a choice of a basis $(f_1,\ldots,f_{n+1})$ in V such that $(f_i,f_j)\in\mathbb{Z}$. Then $M=\mathbb{Z}f_1+\cdots+\mathbb{Z}f_{n+1}$ is a *quadratic lattice*, i.e. a free abelian group equipped with an integral valued non-degenerate quadratic form. We have

$$V\cong M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}.$$

The occurrence of vector spaces with hyperbolic signature in algebraic geometry is explained by the *Hodge Index Theorem*.

Let X be a nonsingular complex projective algebraic variety of dimension d (of real dimension $2d$). Its cohomology $H^k(X,\mathbb{C})$ admits the *Hodge decomposition*

$$H^k(X,\mathbb{C})=\bigoplus_{p+q=k}H^{p,q}(X),$$

where $\overline{H^{p,q}}(X)=H^{q,p}(X)$. The dimensions of $H^{p,q}$) are called the *Hodge numbers* and denoted by $h^{p,q}(X)$. Via the de Rham Theorem, each cohomology class in $H^{p,q}(X)$ can be represented by a complex differential form of type (p,q), i.e. locally expressed in terms of the wedge-products of p forms dz_i and q-forms of type $d\bar{z}_j$. We also have

$$H^{p,q}(X)\cong H^q(X,\Omega_X^p),\tag{1.1}$$

where Ω_X^p is the sheaf of holomorphic differential p-forms on X.

We set

$$H^{p,q}(X,\mathbb{R}) : \; = \; H^{p+q}(X,\mathbb{R}) \cap (H^{p,q}(X) + H^{q,p}(X)), \tag{1.2}$$
$$H^{p,q}(X,\mathbb{Z}) : \; = \; H^{p+q}(X,\mathbb{Z}) \cap (H^{p,q}(X) + H^{q,p}(X)). \tag{1.3}$$

Fix a projective embedding of X in some projective space $\mathbb{P}^N$, then cohomology class η of a hyperplane section of X belongs to $H^{1,1}(X,\mathbb{Z})$.

The cup-product $(\phi,\psi) \mapsto \phi \cup \psi \cup \eta^{d-k}$ defines a bilinear form

$$Q_\eta : H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) \to H^{2d}(X,\mathbb{R}) \cong \mathbb{R},$$

where the last isomorphism is defined by the fundamental class of X.

The *Hard lefschetz Theorem* asserts that, for any $k \geq 0$,

$$L^k : H^{d-k}(X,\mathbb{C}) \stackrel{\phi \mapsto \phi \wedge \eta^k}{\longrightarrow} H^{d+k}(X,\mathbb{C})$$

is an isomorphism compatible with the Hodge structures (i.e. $L^k(H^{p,q}(X)) = H^{p+k,q+k}(X)$). For any $k \geq 0$, let

$$\begin{aligned} H^{d-k}(X,\Lambda)_{\text{prim}} &= \text{Ker}(L^{k+1}), \\ H^{p,q}(X,\Lambda)_{\text{prim}} &= H^{p,q}(X) \cap H^{d-k}(X,\Lambda)_{\text{prim}}, \; p+q = d-k. \end{aligned}$$

where $\Lambda = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. The *Hodge Index Theorem* asserts that the cup-product Q_η satisfies the following properties

- $Q_\eta(H^{p,q}(X), H^{p',q'}(X)) = 0$, if $(p,q) \neq (q',p')$;
- $\sqrt{-1}^{p-q}(-1)^{(d-k)(d-k-1)/2} Q_\eta(\phi,\bar{\phi}) > 0$, for any $\phi \in H^{p,q}(X)_{\text{prim}}, p+q = k$.

Let us apply this to the case when $d = 2m$. In this case, we have the cup-product on the middle cohomology

$$H^d(X,\Lambda) \times H^d(X,\Lambda) \to H^{2d}(X,\Lambda) \cong \Lambda.$$

By *Poincaré Duality* , this is the perfect symmetric duality modulo torsion (of course, no torsion if $\Lambda \neq \mathbb{Z}$)). For $\Lambda = \mathbb{R}$, it coincides with Q_η, and the Hodge Index Theorem asserts in this case that Q_η does not depend on η and its restriction to $H^{m,m}(X,\mathbb{R})_{\text{prim}}$ is a negative (resp. positive) definite symmetric bilinear if $m = 4k, 4k+1$ (resp. $m = 4k+2, 4k+3$). Assume that

$$h^{m-1,m-1}(X) = \mathbb{C}\eta^{m-1} \tag{1.4}$$

Then

$$H^{m,m}(X,\mathbb{R}) = H^{m,m}(X,\mathbb{R})_{\text{prim}} \perp \mathbb{R}\eta^m, \tag{1.5}$$

has signature $(1, h^{m,m}(X) - 1)$. Note that $H^{m,m}(X, \mathbb{R})_{\text{prim}}$ depends on a choice of an embedding $X \hookrightarrow \mathbb{P}^N$, so the previous orthogonal decomposition depends on it too.

We will be interested with hyperbolic vector spaces equipped with an integral structure. For this reason, we should consider the part $H^{p,p}(X, \mathbb{Z})$ of $H^{p,p}(X)$. For any closed algebraic irreducible p-dimensional subvariety Z of X, its fundamental class $[Z]$ belongs to $H^{p,p}(X, \mathbb{Z})$. To see this, we use that, locally on an open subset U, where Z is a complex manifold, we can choose complex coordinates $z_1, \ldots, z_p$. Thus, the restriction of each form ω of type $(p', q') \neq (p, p)$ to U vanishes. This shows that $\int_Z \omega = 0$ for such forms, and hence $[Z]$ defines a cohomology class in $H^{p,p}(X)$. It is obviously integral.

For any $p \geq 0$, let
$$H^{2p}(X, \mathbb{Z})_{\text{alg}} \subset H^{p,p}(X, \mathbb{Z})$$
be the subgroup of cohomology classes generated by the classes $[Z]$, where Z is an irreducible p-dimensional closed subvariety of X. Its elements are called *algebraic cohomology classes*.

The *Hodge Problem* asks whether, for class $\omega \in H^{p,p}(X, \mathbb{Z})$, there exists an integer n such that $n\omega \in H^{2p}(X, \mathbb{Z})_{\text{alg}}$. It is known to be true for $p = 1$ (and n could be taken to be equal to 1). For $p > 1$, it is known only for a few classes of algebraic varieties (see [35]).

We set
$$N^p(X) = H^{2p}(X, \mathbb{Z})_{\text{alg}}/\text{Torsion}.$$
This is a free abelian group of some finite rank $\rho_k(X)$. When $d = 2m = 2(2s+1)$, and X satisfies (1.4), the group $N^m(X)$ defines an integral structure on the hyperbolic vector space
$$N^m(X)_{\mathbb{R}} = N^m(X) \otimes_{\mathbb{Z}} \mathbb{R} \subset H^{m,m}(X, \mathbb{R}).$$
of dimension $1 + \rho_m(X)$.

In the case of algebraic varieties defined over a field $\Bbbk$ different from $\mathbb{C}$ one defines the *Chow group* $\text{CH}^p(X)$ of algebraic cycles of codimension p on X modulo rational equivalence (see [25]). Its quotient $\text{CH}(X)_{\text{alg}}$ modulo the subgroup of cycles algebraically equivalent to zero is the closest substitute of $H^{2p}(X, \mathbb{Z})_{\text{alg}}$.

The intersection theory of algebraic cycles defines the symmetric intersection product
$$\text{CH}^k(X) \times \text{CH}^l(X) \to \text{CH}^{k+l}(X),\ (\gamma, \beta) \mapsto \gamma \cdot \beta$$
It descents to the intersection product
$$\text{CH}^k(X)_{\text{alg}} \times \text{CH}^l(X)_{\text{alg}} \to \text{CH}^{k+l}(X)_{\text{alg}}.$$
When $\Bbbk = \mathbb{C}$, there is a natural homomorphism $\text{CH}^p(X)_{\text{alg}} \to H^{p,p}(X, \mathbb{Z})$, however it may not be injective if $p > 1$.

The group $\text{CH}^1(X)$ coincides with the *Picard group* $\text{Pic}(X)$ of X, the group of divisors modulo linear equivalence. It is naturally identified with the group of isomorphism classes of line bundles (or invertible sheaves) on X (see [30]). The group $\mathrm{CH}^1(X)_{\mathrm{alg}}$ is denoted by $\mathrm{NS}(X)$ and is called the *Néron-Severi group* of X.

One defines the group $N^p(X)$ of *numerical equivalence classes* of algebraic cycles as the quotient group of $\mathrm{CH}^p(X)$ modulo the subgroup of elements γ such that $\gamma \cdot \beta = 0$ for all $\beta \in \mathrm{CH}^{d-p}(X)$. It is not known whether this definition coincides with the previous definition when $\Bbbk = \mathbb{C}$ and $p > 1$.

The statement about the signature of the intersection product on $N^d(X)_{\mathbb{R}}$ is a conjecture. It follows from *Standard Conjectures* of A. Grothendieck (see [32]).

The group $N^1(X)$ coincides with the group $\mathrm{Num}(X)$ of numerical classes of divisors on X. It is the quotient of the Néron-Severi group by the torsion subgroup.

Example 1.1. Assume $d = 2$, i.e. X is a nonsingular projective algebraic surface. Since $H^0(X, \Lambda) \cong \Lambda$, we have $h^{0,0} = 1$, and condition (1.4) is obviously satisfied. In this case

$$N^1(X) = H^{1,1}(X, \mathbb{Z})/\text{Torsion}.$$

The number $\rho_1(X)$ is called the *Picard number* of X and is denoted by $\rho(X)$. The Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

has the Hodge numbers $h^{2,0} = h^{0,2} = \dim H^0(X, \Omega_X^2)$. By Serre's Duality, $\dim_{\mathbb{C}} H^0(X, \Omega_X^2) = \dim_{\mathbb{C}} H^2(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of regular (=holomorphic) functions on X. This number is classically denoted by by $p_g(X)$ and is called the *geometric genus* of X. We have

$$\rho(X) \leq h^{1,1}(X) = b_2(X) - 2p_g(X), \tag{1.6}$$

where $b_k(X)$ denote the Betti numbers of X.

Note that, in the case of surfaces, the Hodge Index Theorem can be proved without using the Hodge decomposition, and it is true for arbitrary fields. The group $N^1(X)$ is defined to be the group of linear equivalence classes of divisors on X modulo divisors numerically equivalent to zero, i.e. divisors D such that $D \cdot D'$ with any divisor class D' on X is equal to zero (see [30], Chapter 5, Theorem 1.9). The Lorentzian vector space in this case is $N^1(X)_{\mathbb{R}}$.

Example 1.2. Let X be a complete intersection in $\mathbb{P}^{k+d}$. This means that X is defined in $\mathbb{P}^{d+k}$ by k homogeneous equations of some degrees $(a_1, \ldots, a_k)$. We assume, as before, that $d = 2m$, where m is odd. The *Lefschetz Theorem* on a hyperplane section asserts that the restriction homomorphism

$$H^i(\mathbb{P}^{d+k}, \mathbb{Z}) \to H^i(X, \mathbb{Z})$$

is bijective for $i \leq d - 2$. This implies that

$$H^{2i}(X, \mathbb{Z}) = \mathbb{Z}\eta^i, \ i < d,$$

where η is the class of a hyperplane section. Thus the condition (1.4) is satisfied, and we conclude that $N^m(X)$ defines an integral structure on the Lorentzian vector space $N^m(X)_{\mathbb{R}}$.

Another example of occurrence of vector spaces with a hyperbolic signature in algebraic geometry is provided by the *Beauville-Bogomolov quadratic form* on the cohomology of a holomorphic symplectic variety X.

A *holomorphic symplectic complex connected manifold* is a compact complex Kähler manifold X such $H^0(X,\Omega_X^2)$ is generated by a nowhere vanishing holomorphic 2-form α (a *holomorphic symplectic form*). On each tangent space T_xX the form defines a non-degenerate skew-symmetric 2-form, hence $\dim X = \dim T_xX = 2n$. The wedge nth power of α is a nowhere vanishing $2n$-form, hence $K_X = 0$. By Bogomolov's Decomposition Theorem, any simply-connected compact Kähler manifold M with $K_M = 0$ is isomorphic to the product of simply-connected holomorphic symplectic manifolds with $H^0(X,\Omega_X^*) = \mathbb{C}[\omega]$ and Calabi-Yau manifolds satisfying $H^0(M,\Omega_M^k) = 0, 0 < k < \dim M$.

Definition 1.1. An *irreducible holomorphic symplectic manifold* is a holomorphic symplectic complex connected manifold satisfying $H^0(X,\Omega_X^*) = \mathbb{C}[\omega]$.

In particular, such a manifold satisfies $b_1(X) = 0$.

The second cohomology group $H^2(X,\mathbb{R})$ of an irreducible holomorphic symplectic manifold of dimension $2n$ is equipped with a quadratic form q_{BB}, the *Beauville-Bogomolov form*: It is defined by the property that

$$q_{\mathrm{BB}}(\gamma) = \tfrac{1}{2}n\int_X \gamma^2(\alpha\wedge\bar{\alpha})^{n-1} + (1-n)\Big(\int_X \gamma\alpha^n\wedge\bar{\alpha}^{n-1}\Big)\Big(\int_X \gamma\alpha^{n-1}\wedge\bar{\alpha}^n\Big),$$

where α is a nonzero holomorphic 2-form on X normalized by the condition that $\int_X \alpha^n\wedge\bar{\alpha}^n = 1$.

We have

$$\gamma^n = c q_{\mathrm{BB}}(\gamma)^n,$$

for some constant c (called the *Fujiki constant*) such that q_{BB} defines a primitive quadratic form on $H^2(X,\mathbb{Z})$ with values in $\mathbb{Z}$. The quadratic form q_{BB} is of signature $(3, b_2-3)$ and is invariant with respect to deformations. The Hodge decomposition $H^2(X,\mathbb{C}) = H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$ is an orthogonal decomposition with respect to the Beauville-Bogomolov form q_{BB}. Its signature on $H^{1,1}(X,\mathbb{R})$ is $(1, b_2-3)$. It has an integral structure defined by $H^2(X,\mathbb{Z})$.

In the case $n = 2$, an irreducible holomorphic symplectic manifold is a Kähler *K3 surface* (all K3 surfaces are Kähler). We will discuss these surfaces later.

Example 1.3. Let $X : F_3(t_0,\dots,t_5) = 0$ be a nonsingular cubic hypersurface in $\mathbb{P}^5$. The set of lines contained in it is a nonsingular 4-dimensional subvariety $F(X)$ of the Grassmann variety of lines in $\mathbb{P}^5$. Let

$$Z = \{(x,\ell)\in X\times F(X) : x\in\ell\}$$

be the incidence variety. It comes with two projections

$$\begin{array}{ccc} & Z & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & F(X) \end{array}$$

The projection $p : Z \to F(X)$ is a $\mathbb{P}^1$-bundle and the projection $q : Z \to X$ is a fibration with a general fiber isomorphic to the set of lines passing through a general point in X. It can be shown that these lines are naturally parameterized by a nonsingular curve of genus 4, but this does not matter for the following. The correspondence defines a homomorphism of Hodge structures

$$\Phi = p_* q^* : H^*(X, \mathbb{C})[2] \to H^*(F(X), \mathbb{C}),$$

where $[2]$ means that $H^{p,q}$ are mapped to $H^{p-1,q-1}$. In particular,

$$H^4(X), \mathbb{C} = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X) \cong H^{2,0}(F(X)) \oplus H^{1,1}(F(X)) \oplus H^{0,2}(F(X)).$$

One computes $h^{3,1}(X) = 1, h^{2,2}(X) = 21$. The image of a generator of $H^{3,1}(X)$ defines a holomorphic symplectic form on $F(X)$. This defines a structure of an irreducible holomorphic symplectic manifold on the variety of lines of X.

The variety $F(X)$ comes with a natural Plücker projective embedding of the Grassmannian. Let η be the class of a hyperplane in $N^1(X)$. For a general cubic 4-fold $N^1(X) = \mathbb{Z}\eta$. However, when X specializes the Picard number $\rho_1(F(X)$ could take all possible values between 1 and 20. So, we can realize some of discrete subgroups in $\mathbb{H}^n, n \leq 19$, as automorphism groups of $F(X)$. Note that some of the automorphisms come from automorphisms of the cubic 4-fold X. However, since, by example 1.2, $H^2(X, \mathbb{Z})$ is generated by the class of a hyperplane section of X in $\mathbb{P}^5$, any automorphism of X is a projective automorphism and $\mathrm{Aut}(X)^*$ is a finite group (one can also show that $\mathrm{Aut}(X)^0$ is trivial). So, interesting automorphisms do not come from symmetries of X.

Lecture 2

Hyperbolic space $\mathbb{H}^n$

A hyperbolic n-dimensional space (or a *Lobachevsky space*) is a n-dimensional Riemannian simply connected space $\mathbb{H}^n$ of constant negative curvature. The following are standard models of this space (see [1]).

Let V be a Minkowski vector space of dimension $n+1$ with the quadratic form $q : V \to \mathbb{R}$ of signature $(1, n)$. We denote the corresponding symmetric bilinear form by (x, y). Let

$$V^+ = \{v \in V : q(v) > 0\}$$

be the interior of the real quadric $q = 0$ in V. The *Lobachevsky (or projective) model* of $\mathbb{H}^n$ is the image of $\mathbb{P}V^+$ in the real projective space $\mathbb{P}(V)/\mathbb{R}^*$. Choose an orthogonal decomposition of V as the orthogonal sum $\mathbb{R}v_0 \oplus V_1$, where $q(v_0) = 1$. The projection to V_1 defines a bijection from $\mathbb{P}V^+$ to the unit real n-dimensional ball

$$B_n = \{v \in V_1 : q(v) < 1\}.$$

Choose the coordinates $t_0, \ldots, t_n$ in V such that $q = t_0^2 - t_1^2 - \cdots - t_n^2$ (we call such coordinates *standard coordinates*). The space V will be identified with the standard Lorentzian space $\mathbb{R}^{1,n}$. If we pass to affine coordinates by setting $x_i = t_i/t_0$, then

$$B_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\}.$$

The model of $\mathbb{H}^n$ equal to $\mathbb{V}^+$, or, equivalently, B_n is called the *projective or the Klein model*. The distance between two points $d(x, y)$ in this model is equal to

$$d(x, y) = \frac{1}{2}|\ln R(a, x, y, b)|,$$

where $R(a, b, xy)$ is the cross ratio of the four points (a, x, y, b) on the line in $\mathbb{P}(V)$ joining the points x, y, where the points a, b are the *ends* of the line, i.e. the points where the line intersects the quadric $q = 0$. We also assume that the points are ordered $a < x < y < b$. It does not spoil the symmetry of the distance $d(x,y) = d(y,x)$. The Riemannian metric in coordinates $(x_1,\dots,x_n)$ in B_n is equal to

$$ds^2 = \frac{(1-|x|^2)\sum_{i=1}^n dx_i^2 + (\sum_{i=1}^n x_i dx_i)^2}{(1-|x|^2)^2},$$

where $|x|^2 = \sum_{i=1}^n x_i^2$.

Another model is obtained by representing points in $\mathbb{P}V^+$ by vectors v with norm $q(v) = 1$. In the standard coordinates, it is a 2-sheeted hyperboloid. We further fix one of its connected components, say defined by $t_0 > 0$. This model is called a *vector model* of $\mathbb{H}^n$.

The distance in the vector model is defined by

$$\cosh d(x,y) = (x,y). \tag{2.1}$$

The Riemannian metric in this model is induced by the standard constant hyperbolic metric $-dx_0^2 + dx_1^2 + \cdots + dx_n^2$ on V.

The third model is the *Poincaré model* or the *conformal model*. As a set, it is still the ball B_n as above. However, the metric is different. It is given by

$$ds^2 = 4(1-|u|^2)^{-2}\sum_{i=1}^{n-1} du_i^2.$$

The isomorphism of Riemannian spaces from the Klein model B_n to the Poincaré model $'B_n$ is defined by the composition of the projection from B_n to the southern hemisphere of the boundary of $\bar{B}_{n+1}$ and then the stereographic projection from the southern pole to the Euclidean plane.

Recall that a *geodesic line* in a Riemannian manifold M is a continuous map $\gamma : \mathbb{R} \to M$ such that $d(\gamma(a), \gamma(b)) = |a-b|$. We also refer to the image of γ as a geodesic (unparameterized) line. For any two distinct points $x, y \in M$ there exists a closed interval $[a,b]$ and a geodesic γ with $\gamma(a) = x, \gamma(b) = y$. It is called the *geodesic segment*. In $\mathbb{H}^n$, such a geodesic is unique and realizes the shortest curve arc from x to y.

For example, the geodesic lines in the Klein model of $\mathbb{H}^2$ are non-empty intersection of lines in $\mathbb{R}^2$ with the interior of the unit disc.

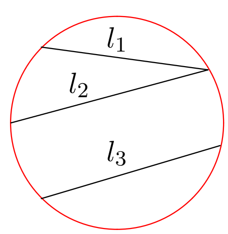


Figure 2.1: Lines on hyperbolic plane in Klein model

Here the lines l_1 and l_2 are parallel and the lines $l_1(l_2)$ and l_3 are *divergent*.

Finally, we have the *upper half plane model* of $\mathbb{H}^n$. It is equal to the Cartesian product $\mathbb{R}^{n-1}\times\mathbb{R}^+$. If we take the coordinates $(w_1,\ldots,w_{n-1},w_n)$ with $w_n>0$, then the bijection from this model to the Klein model is given by the formula

$$y_i=\frac{2x_i}{\rho^2},\ i=1,\ldots,n-1,\ y_n=\frac{1-|x|^2}{\rho^2}, \tag{2.2}$$

where $\rho^2=|x|^2+2x_n+1$. In the case $n=2$, this is the standard biholomorphic mapping from the unit disk $|z|<1$ to the upper-half plane.

The Riemannian metric on the upper-half space model is given by

$$ds^2=y_n^{-2}\sum_{i=1}^n dy_i^2.$$

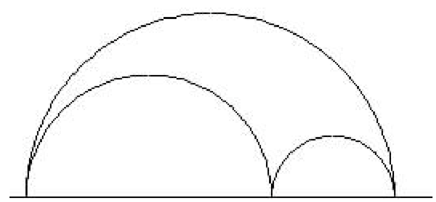


Figure 2.2: Lines on hyperbolic plane in the upper-half plane model

Let us summarize the formulas for the isometries between the four models of $\mathbb{H}^n$.

V-K :

$$(t_0,\ldots,t_n)\mapsto(x_1,\ldots,x_n)=(t_1/t_0,\ldots,t_n/t_0).$$

K-V :

$$(x_1,\ldots,x_n)\mapsto\left(\frac{1}{1-|x|^2},\frac{x_1}{1-|x|^2},\ldots,\frac{x_n}{1-|x|^2}\right).$$

K-P :

$$(x_1,\ldots,x_n)\mapsto(u_1,\ldots,u_n)=\left(\frac{x_1}{1+(1-|x|^2)^{1/2}},\ldots,\frac{x_n}{1+(1-|x|^2)^{1/2}}\right).$$

P-K :

$$(u_1,\ldots,u_n)\mapsto(x_1,\ldots,x_n)=\left(\frac{2u_1}{1+|u|^2},\ldots,\frac{2u_n}{1+|u|^2}\right).$$

P-U :

$$(u_1,\ldots,u_n)\mapsto(y_1,\ldots,y_n)=\left(\frac{2u_i}{\rho^2},\ldots,\frac{2u_{n-1}}{\rho^2},\frac{1-|u|^2}{\rho^2}\right),\ \rho^2=|u|^2+(2u_n+1).$$

U-P :

$$(y_1,\ldots,y_n) \mapsto (u_1,\ldots,u_n) = (\frac{2y_i}{\rho^2},\ldots,\frac{2y_{n-1}}{\rho^2},\frac{|y|^2-1}{\rho^2}),\ \rho^2 = |y|^2+(2y_n+1).$$

Here V, K, P, U stands for the vector, Klein, Poincaré and upper-half space models.

When $n = 2$, the Klein model, Poincaré and the upper-half plane models acquire a structure of a complex 1-manifold with coordinates $z = a + ib, u = u_1 + iu_2, y = y_1 + iy_2$, respectively. The first two models are the unit disks $|z| < 1, |u| < 1$ and the upper-half model is defined by $y_2 > 0$. We have

$$y = i\frac{z+1}{z-1},\ z = \frac{y-i}{y+i},\ z = \frac{2u}{1+|u|^2},\ u = z\frac{1-\sqrt{1-|z|^2}}{|z|^2}.$$

Note that the last two maps are a not biholomorphic but just conformal maps. They relate the the Klein and Poincaré metrics in the unit disk.

We already know how the geodesic lines look like in the Klein model. Let us see how do they look in the Poincaré model. First we assume that a line is given by equation $y_1 = c$, where $|c| < 1$. Assume $c \neq 0$. Then, using the previous formulas, we obtain that the image of this line in the Poincaré model is the circle

$$(u_1 - c^{-1})^2 + u_2^2 = c^{-2} - 1.$$

Its center $(c^{-1}, 0)$ lies outside the (open) unit disk. It intersects the boundary $u_1^2 + u_2^2 = 1$ at two points $(\pm\sqrt{1-\frac{c^2}{4}}, \frac{c}{2})$ and it is orthogonal to the boundary at these points. In the special case when $c = 0$, the image is the diameter line $u_1 = 0$ of the disk. Now, any line $\overline{a,b}$ with the boundary points $a = e^{2i\phi}, b = e^{2i\psi}$ is equal to the image of the vertical line $u_1 = c$ under the transformation $z \mapsto e^{-(\psi+\phi)}z$. It is easy to see that, under the conformal bijection between the two disks, this transformation is also a rotation. This shows that the image of a general line is an arc of a circle orthogonal to the boundary of the segment of a line passing through the origin. This gives a description of geodesic lines in the Poincaré model of $\mathbb{H}^2$. Here the lines are parallel if their closure intersect at one point on the absolute.

The closure in its projective model is the set

$$\overline{\mathbb{H}}^n = \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i^2 \geq 0\}/\mathbb{R}^* \subset \mathbb{P}^n(\mathbb{R}).$$

Its boundary is the real quadric in $\mathbb{P}^n(\mathbb{R})$

$$\partial\mathbb{H}^n = \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i^2 = 0\}/\mathbb{R}^* \subset \mathbb{P}^n(\mathbb{R}).$$

Following F. Klein, it is called the *absolute* of $\mathbb{H}^n$. In the Klein or Poincaré models, the absolute is the $(n-1)$-dimensional sphere, the boundary of the open n-ball. In the usual way, we may consider

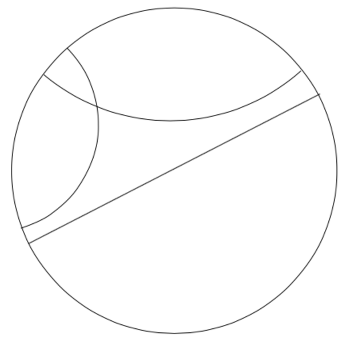


Figure 2.3: Lines on hyperbolic plane in the Poincaré model

the absolute as a one-point compactification of the Euclidean space E^{n-1}. This can be achieved by the stereographic projection from the north pole $p = [1, 1, \dots, 0] \in \partial\mathbb{H}^n$ to the hyperplane $t_0 = 0$ identified with $\mathbb{P}^{n-1}(\mathbb{R})$ with projective coordinates $t_1, \dots, t_n$.[1] For any point $x = [a_0, \dots, a_n] \neq p$, its projection is equal to the point $[a_0 - a_1, a_2 \dots, a_n] \in \mathbb{P}^{n-1}(\mathbb{R})$. Since $(a_0 - a_1)(a_0 + a_1) = a_2^2 + \cdots + a_n^2$, the first coordinate $a_0 - a_1$ is not equal to zero. Thus the projection of $\partial\mathbb{H}^n \setminus \{p\}$ is contained in the affine set $t_1 \neq 0$ which we can identify with the Euclidean space $\mathbb{R}^n$. It is easy to see that it coincides with this affine set, so $\partial\mathbb{H}^n$ can be identified with $\hat{E}^n := E^n \cup \{p\}$.

[1] we use the notation $[a_0, \dots, a_n]$ for the projective coordinates in $\mathbb{P}^n(\mathbb{R})$, other standard notation is $(a_0 : a_1 : \dots : a_n)$.

Lecture 3

Motions of a hyperbolic space

A *motion* or an *isometry* of a Riemannian manifold is a smooth map of manifolds that preserves the Riemannian metric. In our case the group of motions consists of projective transformations that preserve the projective model of $\mathbb{H}^n$. When we choose the standard coordinates this group is the projective orthogonal group $\mathrm{PO}(1,n)$. We denote it by $\mathrm{Iso}(\mathbb{H}^n)$. Note that $\mathrm{PO}(1,n)$ is isomorphic to the index 2 subgroup $\mathrm{O}(1,n)'$ of $\mathrm{O}(1,n)$ that preserves the connected components of the 2-sheeted hyperboloid $\{x \in \mathbb{R}^{1,n} : q(x) = 1\}$. The group $\mathrm{Iso}(\mathbb{H}^n)$ is a real Lie group, it consists of two connected components. The component of the identity $\mathrm{Iso}(\mathbb{H}^n)^+$ consists of motions preserving an orientation.

A discrete subgroup Γ of $\mathrm{Iso}(\mathbb{H}^n)^+$ is called a *Kleinian group*. Sometimes this name is reserved only for subgroups of $\mathrm{Iso}(\mathbb{H}^3)^+$. In this case, $\mathrm{Iso}(\mathbb{H}^3)^+$ is isomorphic to $\mathrm{PSO}(1,3)$. The double cover of this group is the group $\mathrm{Spin}(1,3)$ isomorphic to $\mathrm{SL}(2,\mathbb{C})$, so a Kleinian group becomes isomorphic to a discrete subgroup of the group $\mathrm{PSL}(2,\mathbb{C})$ of Moebius transformations $z \mapsto \frac{az+b}{cz+d}$ of the complex projective line $\mathbb{P}^1(\mathbb{C})$.

Let Γ be a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$. It acts on $\mathbb{H}^n$ totally discontinuously, i.e., for any compact subset K, the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite. In particular, the stabilizer of any point is a finite group. The action extends to the absolute, however here Γ does not act discontinuously. We define the *limit set* $\Lambda(\Gamma)$ to be the complement of the maximal open subset of $\partial\mathbb{H}^n$ where Γ acts discretely. It can be also defined as the closure of the orbit $\Gamma \cdot x$ for any $x \in \mathbb{H}^n$. It is also equal to the closure of the set of fixed points on $\partial\mathbb{H}^n$ of elements of infinite order in Γ. For any $\gamma \in \Gamma$ there are three possible cases:

- γ is *hyperbolic*: there are two distinct fixed points of γ in $\partial\mathbb{H}^n$;
- γ is *parabolic*: there is one fixed point of γ in $\partial\mathbb{H}^n$;
- γ is *elliptic*: γ is of finite order and has a fixed point in $\mathbb{H}^n$;

Let γ be a hyperbolic isometry. Then there exists a hyperbolic plane U in V generated by isotropic vectors u, v with $(u, v) = 1$ such that $\gamma(U) = U$ and $\gamma|U$ is given by the matrix $A(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{\pm 1}$ for some $\lambda > 1$. The fixed points of γ on $\partial\mathbb{H}^n$ are represented by the vectors u and v. The distance between two points $x = [e^t u + e^{-t} v], y = [e^{t'} u + e^{-t'} v]$ represented by vectors with norm 1 on the geodesic $\gamma = \mathbb{P}(U) \cap \mathbb{H}^2 \subset \mathbb{H}^n$ is equal to $\frac{1}{2} \ln |R|$, where R is the cross-ratio of the points $(0, x, y, \infty)$. It is equal to $e^{2(t-t')}$, so we get $d(x, y) = |t - t'|$. So, we see that t is the natural parameter on the geodesic γ, and the isometry γ_λ with $|\lambda| = |t - t'|$ moves x to the point $\gamma(x)$ with $d(x, \gamma(x)) = |\lambda|$. It also shows that one can identify the geodesic γ with the one-parameter subgroup $\{g_t\}_{t \in \mathbb{R}}$ defined by the matrix $e^{A(\lambda)t}$. It is called the *axis* of g. Also, we see that, any geodesic is a geodesic in some 2-dimensional hyperbolic subspace of $\mathbb{H}^n$.

Let γ be a parabolic isometry. Then there exists an isotropic vector u such that $[u] \in \partial\mathbb{H}^n$ is fixed by γ. Then γ leaves invariant $u^\perp$ and acts naturally on $u^\perp/\mathbb{R}u \cong \mathbb{R}^{0,n-1}$. One can choose a hyperbolic plane U as above with a basis u, v and a vector $w \in U^\perp$ such that γ leaves invariant $U \oplus \mathbb{R}w$ and is given in the basis (u, w, v) by the matrix $\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Let us take a one-parameter subgroup generated by a parabolic isometry γ. It consists of transformations γ_t given by the matrices $\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$. It is clear that γ acts in $u^\perp/\mathbb{R}u$ that we can identify with $U^\perp \cong R^{0,n-1}$. If we change the sign of the quadratic form, this becomes the Euclidean vector space. It embeds in $\mathbb{H}^n$ via the map $[x] \mapsto [x + cv + u]$. The image is the intersection of $\mathbb{H}^n$ with the hyperplane $H_u(c) = \{x \in V : (x, u) = c\}/\mathbb{R}^*$. It is called a *horosphere* in $\mathbb{H}^n$. Its closure in $\overline{\mathbb{H}}^n$ contains the points $[x + cv + u], (x, x) = -2c$ in the absolute. This is the image of a sphere in the Euclidean space $u^\perp/\mathbb{R}u$ of radius $r = \sqrt{2c}$. For example, when $n = 2$ and we have a basis (u, w, v) as above, the closure of the horosphere $H_u(c)$ intersects the boundary at one point $[u + cv + \sqrt{-2c}w]$. In the Klein model, if we take $u = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0], v = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0], w = [0, 0, 1]$, then $[u] = (1, 0), [v] = (-1, 0), [w] = (0, 0)$ in the closed disk. The horosphere becomes the set of points $(\frac{1-c}{1+c}, \frac{\alpha}{1+c})$, i.e. the vertical line $z_1 = \frac{1-c}{1+c}$, where $\alpha^2 < 1 + 2c$. Its image in the Poincaré plane is the set $\{(u_1, u_2) : 2u_1 = \frac{1-c}{1+c}(1 + u_1^2 + u_2^2)\}$. Its closure is the circle

$$(u_1 - a)^2 + u_2^2 = a^2 - 1,\ a = \frac{1-c}{1+c}.$$

Its center lies outside the unite disk, and it is tangent to the boundary at the point with $u_1 = \frac{1-c}{1+c}$. All horospheres (or rather *horocircles*) are circles tangent to the boundary at one point. The orbits of one parameter subgroups are horospheres.

A geodesic line in $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a line in $\mathbb{P}^n(\mathbb{R})$ corresponding to a hyperbolic plane P. in $\mathbb{R}^{1,n}$. Consider a pencil $\mathcal{P}_e$ of lines passing through a fixed point $[e] \in \mathbb{P}^n(\mathbb{R})$.

Let $[v] \in \mathbb{H}^n$ be a point on a line from $\mathcal{P}_e$ and $c = (v, e)$. Then the image $H_e(c)$ of the affine hyperplane $\{x \in \mathbb{R}^{1,n} : (x, e) = c\}$ in $\mathbb{P}^n(\mathbb{R})$ is equal to the image of the translate $H_e(0) + v$ of the homogeneous hyperplane $H_e(0)$. We have $H_e(c) = H_{\lambda e}(\lambda c)$. Its normal vector is e and it is orthogonal to all vectors $x - v \in H_e(c)$. If $[e] \in \mathbb{H}^n$ and $[v] \in H_e(c)$, we may assume that $(e, e) = 1, (v, v) = 1, (e, v) = c$ (we must have $c > 1$ in this case) then $\cosh d(e, v) = (e, v) = c$, so we may think that the hyperplane $H_e(c)$ is the locus of points in $\mathbb{H}$ with hyperbolic distance from

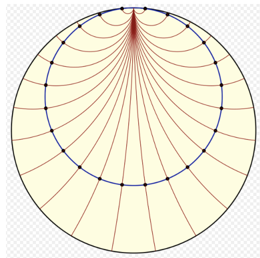


Figure 3.1: Parabolic pencil in the Poincaré model

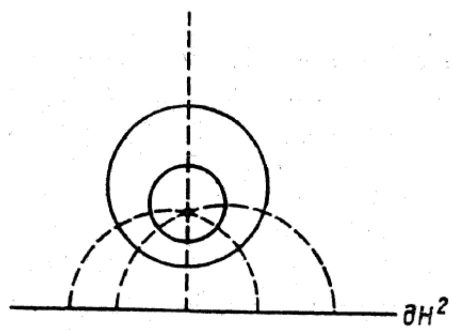


Figure 3.2: Elliptic pencil in the upper-half plane model

$[e]$ equal to c. It is *hyperbolic ball* $\mathcal{B}_e(c)^+$ of radius c and center $[e]$. The pencil $\mathcal{P}_e$ in this case is called *elliptic pencil of lines*.

If $(e, e) < 0$, then $H_e(0) \cap \mathbb{H}^n \neq \emptyset$ and $\mathcal{P}_e$ consists of lines perpendicular to $H_e(0)$. The hyperplanes $H_e(c)$ are also perpendicular to all lines in the pencil. This is a *hyperbolic pencil of lines*. If $[x] \in H_e(c)$, we can define the distance from $[e]$ to $[x]$ as the distance from $[x]$ to the hyperplane $H_e(0)$. Thus $H_e(c) \cap \mathbb{H}^n$ can be viewed as the set of all points $x \in \mathbb{H}$ such that the distance from x to $[e]$.

Finally, if $(e, e) = 0$, Then $H_e(c)$ is invariant with respect to the translations $[x] \mapsto [x + \lambda e] = [\lambda^{-1}x + e]$. So, $H_e(c)$ is invariant with respect to parabolic transformation which fix $[e]$. It is a horosphere with center at $[e]$. When λ goes to zero, the limit point is equal to $[e]$. So, the closure of $H_e(c)$ is the point $[e]$ on the absolute. At each intersection point, a line from the pencil $\mathcal{P}_e$ is perpendicular to $H_e(c)$. The pencil in this case is called a *parabolic pencil of lines*. Fix e by choosing another isotropic vector e' with $(e, e') = 1$, and then fix the horosphere $H_e(1)$. We define the distance between $[e]$ and $[x] \in \mathbb{H}^n$ as the distance from $[x]$ to the horosphere $H_e(1)$. This leads to the definition of a ball $\mathcal{B}_e(c)^0$.

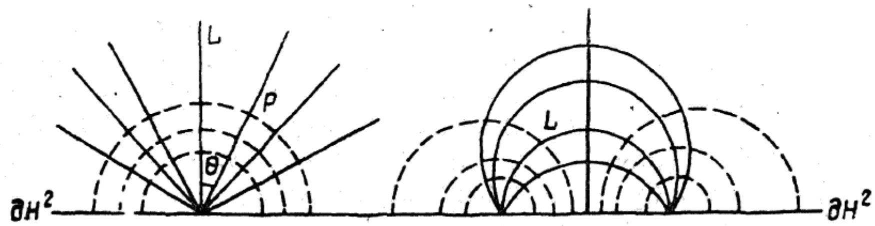


Figure 3.3: Hyperbolic pencil in the upper-half plane model

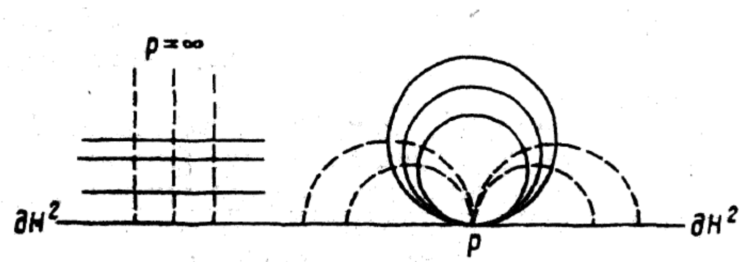


Figure 3.4: Parabolic pencil in the upper-half plane model

Proposition 3.1. *Let* $[e] \in \mathbb{P}^n(\mathbb{R})$ *and* $[x] \in \mathbb{H}^n$. *Then*

$$(x,e) = \begin{cases} \cosh d([x],[e]) & \textit{if}\,,\ (e,e) = (x,x) = 1, \\ \sinh d([x],[e]) & \textit{if}\,,\ (e,e) = -1, (x,x) = 1, (x,e) = 0, \\ \exp(d(x,[e])) & \textit{if}\ (e,e) = 0, (x,x) = 1, (x,e) = -1. \end{cases}$$

Proof. If $[e] \in \mathbb{H}^n$, this follows from the formula for the distance in the vector model of $\mathbb{H}^n$. If $(e,e) = -1$, we parameterize a line joining $[e]$ with $[x]$ as $[-e\sinh(t)+\cosh(t)x]$, where t is the natural (distance) parameter on the line ([1], Chapter 4,2.2.3). We assume that it measures the distance from the point in the intersection of the line with $H_e(0)$. This shows that $(x,e) = \sinh d([e],[x])$. Similarly, if $(e,e) = 0$, we parameterize a line joining $[e]$ with $[x]$ as $[-e\sinh(t) + \exp(t)x]$, where t is the natural parameter which measures the distance from the point in the intersection of the line with the horosphere $H_e(1)$. This gives $\exp(d([e],[x]) = (x,e)$. □

A non-empty subset A of a metric space is called *convex* if any geodesic connecting two of its points is contained in A. Its closure is called a *closed convex subset*. A closed convex subset P of $\mathbb{H}^n$ is equal to the intersection of half-spaces $H^{\pm}_{\mathfrak{e}_i}, i \in I$. It is assumed that none of the half-spaces $H_{\mathfrak{e}_i}$ contains the intersection of other half-spaces. A maximal non-empty closed convex subset of the boundary of P is called a *side*. Each side is equal to the intersection of P with one of the bounding hyperplanes $H_{\mathfrak{e}_i}$. The boundary of P is equal to the union of sides and two different sides intersect only along their boundaries.

A *convex polyhedron* in $\mathbb{H}^n$ is a convex closed subset P with finitely many sides. It is bounded by a finite set of hyperplanes H_i.

We choose the vectors $\mathfrak{e}_i$ as in the previous section so that

$$P = \bigcap_{i=1}^{N} H_{\mathfrak{e}_i}^{-}. \tag{3.1}$$

The *dihedral angle* $\phi(H_{\mathfrak{e}_i}, H_{\mathfrak{e}_j})$ between two proper bounding hyperplanes is defined by the formula

$$\cos\phi(H_{\mathfrak{e}_i}, H_{\mathfrak{e}_j}) := -(\mathfrak{e}_i, \mathfrak{e}_j).$$

If $|(\mathfrak{e}_i, \mathfrak{e}_j)| > 1$, the angle is not defined, we say that the hyperplanes are *divergent*. In this case the distance between $H_{\mathfrak{e}}$ and $H_{\mathfrak{e}'}$ can be found from the formula

$$\cosh d(H_{\mathfrak{e}}, H_{\mathfrak{e}'}) = |(\mathfrak{e}, \mathfrak{e}')|. \tag{3.2}$$

For any subset J of $\{1, \ldots, N\}$ of cardinality $n-k$, such the intersection of $H_{\mathfrak{e}_i}, i \in J$, is a k-plane Π in $\mathbb{H}^n$, the intersection $P \cap \Pi$ is a polyhedron in Π. It is called a *k-face* of P. A $(n-1)$-face of P is a side of P.

The matrix

$$G(P) = (g_{ij}), \quad g_{ij} = (\mathfrak{e}_i, \mathfrak{e}_j),$$

is called the *Gram matrix* of P. There is a natural bijection between the set of its k-dimensional proper faces and positive definite principal submatrices of $G(P)$ of size $n-k$. The improper vertices of P correspond to positive semi-definite principal submatrices of size n.

Recall that a closed subset D of a metric space X is called a *fundamental domain* for a group Γ of isometries of X if

(i) the interior D^o of D is an open non-empty set;

(ii) $\gamma(D^o) \cap D^o = \emptyset$, for any $\gamma \in \Gamma \setminus \{1\}$;

(iii) the set of subsets of the form $\gamma(D)$ is locally finite[1];

(iv) $X = \bigcup_{\gamma \in \Gamma} \gamma(D)$;

A group Γ admits a fundamental domain if and only if it is a discrete subgroup of the group of isometries of X. For example, one can choose D to be a *Dirichlet fundamental domain*

$$D(x_0) = \{x \in X : d(x, x_0) \le d(\gamma(x), x_0), \text{ for any } \gamma \in \Gamma\}, \tag{3.3}$$

[1]This means that every point $x \in \mathbb{H}^n$ is contained in a finite set of subsets $\gamma(D)$

where x_0 is a fixed point in X and $d(x,y)$ denotes the distance between two points. Assume $X = \mathbb{H}^n$. For any $\gamma \in \Gamma \setminus \Gamma_{x_0}$, let H_γ be the hyperplane of points x such that $d(x_0,x) = d(x,\gamma(x_0))$. Then $D(x_0) = \cap_{\gamma\in\Gamma\setminus\Gamma_{x_0}} H_\gamma^-$ and, for any $\gamma \notin \Gamma_{x_0}$, $S = \gamma(D(x_0)) \cap D(x_0) \subset H_\gamma$ is a side in $\partial D(x_0)$. Each side of $D(x_0)$ is obtained in this way for a unique γ.

A fundamental domain D for a Kleinian group Γ in $\mathbb{H}^n$ is called *polyhedral* if its boundary $\partial D = D \setminus D^o$ is contained in the union of a locally finite set of hyperplanes H_i and each side S of the boundary is equal to $D \cap \gamma(D)$ for a unique $\sigma_S \in \Gamma$. A Dirichlet domain is a convex polyhedral fundamental domain.

A choice of a polyhedral fundamental domain allows one to find a presentation of Γ in terms of generators and relations. The set of generators is the set of elements $\gamma_S \in \Gamma$, where S is a side of D. A relation $\gamma_{S_t} \circ \cdots \circ \gamma_{S_1} = 1$ between the generators corresponds to a cycle

$$D_0 = D, D_1 = \gamma_{S_1}(D_0), \ldots, D_t = \gamma_{S_t} \circ \cdots \circ \gamma_{S_1}(D_0) = D_0.$$

Among various equivalent definitions of a geometrically finite Kleinian group we choose the following one (see [2], Chapter 4, §1): A Kleinian group Γ is called *geometrically finite* if it admits a polyhedral fundamental domain with finitely many sides. [2] It follows from above that such a group is finitely generated and finitely presented. The converse is true only in dimension $n \leq 2$. In dimensions $n \leq 3$, one can show that Γ is geometrically finite if and only if there exists (equivalently any) a Dirichlet fundamental domain with finitely many sides (loc. cit., Theorem 4.4). On the other hand, for $n > 3$ there are examples of geometrically finite groups all whose Dirichlet domains have infinitely many sides (loc. cit. Theorem 4.5).

[2] Other equivalent definition is given in terms of the convex core of Γ, the minimal convex subset of $\mathbb{H}^n$ that contains all geodesics connecting any two points in $\Lambda(\Gamma)$.

Lecture 4

Automorphism groups of algebraic varieties

For any projective algebraic variety X over a field $\Bbbk$ one can define a group scheme $\underline{\mathrm{Aut}}(X)$ of automorphisms of X. Its set of $\Bbbk$-points is the group $\mathrm{Aut}(X)$ of automorphisms of X over $\Bbbk$. Over a field of arbitrary characteristic it is a group scheme of locally finite type, not necessary reduced. The connected component of the identity $\underline{\mathrm{Aut}}^0(X)$ is a group scheme of finite type, the reduced scheme is an algebraic group. We denote its set of $\Bbbk$-points by $\mathrm{Aut}(X)^0$. The group

$$\mathrm{Aut}_c(X) := \mathrm{Aut}(X)/\mathrm{Aut}(X)^0$$

is called the *group of connected components* of the automorphism group of X.

Over $\mathbb{C}$, $\mathrm{Aut}(X)$ has a natural structure of a complex Lie group whose connected component of the identity $\mathrm{Aut}(X)^0$ is a complex algebraic group. The group of connected components $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is a countable discrete group..

Let $\mathrm{Aut}(X) = \underline{\mathrm{Aut}}(X)(\mathbb{K})$, where X is defined over an algebraically closed field $\Bbbk$, considered as an abstract group. It has a natural linear representation

$$\alpha_k : \mathrm{Aut}(X) \to \mathrm{GL}(N^k(X)), g \mapsto g^*$$

such that

$$\alpha_k(g)(x) \cdot \alpha_{n-k}(g)(y) = x \cdot y,\ x \in N^k(X), y \in N^{n-k}(X).$$

Here we use the notation $x \cdot y$ for the intersection number of two numerically equivalence classes of algebraic cycles. Over $\mathbb{C}$ it coincides with the cup-product of the corresponding cohomology classes. In particular, when $d = 2m$, this defines a homomorphism

$$\alpha : \mathrm{Aut}(X) \to \mathrm{O}(N^m(X)),$$

where $\mathrm{O}(N^m(X))$ is the orthogonal group of the quadratic lattice $N^m(X)$. The group $\underline{Aut}^0(X)(\mathbb{K})$ sits in the kernel of α, so the homomorphism factors through the group $\mathrm{Aut}_c(X)$ of connected components of $Aut(X)$ and defines a homomorphism

$$\alpha_c : \mathrm{Aut}_c(X) \to \mathrm{O}(N^m(X)).$$

One can show that the kernel of this homomorphism is a finite group. We denote by $\mathrm{Aut}(X)^*$ the image of α in $\mathrm{O}(N^m(X))$.

Recall that any nonsingular algebraic variety X of dimension d carries a distinguished divisor class in the Picard group $\mathrm{Pic}(X)$, the *canonical class* K_X. Over $\mathbb{C}$, it is a distinguished cohomology class in $H^{1,1}(X,\mathbb{Z})$ equal to the negative of the *first Chern class* $c_1(X)$ of the holomorphic tangent bundle of X. Under the identification of the Picard group with the group of isomorphism classes of invertible sheaves, it corresponds to the sheaf Ω_X^d of differential d-forms.

The canonical class K_X is invariant with respect to any automorphism of X, i.e. $g^*(K_X) = K_X$, for any $g \in \mathrm{Aut}(X)$. If $d = 2m$, the m-th self-product $K_X^m := K_X \cdots K_X$ is an invariant class in the group of algebraic cycles CH^m. We denote by k_X its image in $N^m(X)$ (or in $H^d(X,\mathbb{Z})$ when $\Bbbk = \mathbb{C}$). The group $\mathrm{Aut}(X)^*$ stabilizes the vector k_X, and, hence

$$\mathrm{Aut}(X)^* \subset \mathrm{O}(N^m(X))_{k_X}.$$

Suppose $k_X \neq 0$, let $k_X^\perp$ denote the orthogonal complement of $\mathbb{Z}k_X$ in $N^m(X)$. Then we can identify $\mathrm{Aut}(X)^*$ with its image under the natural homomorphism $\mathrm{O}(N^m(X))_{k_X} \to \mathrm{O}(k_X^\perp)$. That allows us to consider $\mathrm{Aut}(X)^*$ as a subgroup of $\mathrm{O}(k_X^\perp)$.

Theorem 4.1. *Assume $N^m(X)_\mathbb{R}$ has hyperbolic signature $(1,n)$, i.e. $d \equiv 2 \mod 4$. Then $\mathrm{Aut}(X)^*$ is a finite subgroup of $\mathrm{Iso}(\mathbb{H}^n)$ if $k_X^2 > 0$ and an elementary geometrically finite group if $k_X^2 = 0$.*

Proof. Suppose $k_X^2 > 0$, then $k_X^\perp$ is a negative definite quadratic lattice. The group $\mathrm{Aut}(X)^*$ is a discrete subgroup of the compact Lie group $\mathrm{O}(k_X^\perp) \cong \mathrm{O}(n)$, so it is finite.

If $k_X^2 = 0$, the lattice $R = k_X^\perp/\mathbb{Z}k_X$ is negative definite and $\mathrm{O}(k_X^\perp)$ is isomorphic to the semi-direct product $R \rtimes \mathrm{O}(R)$. It contains a free finitely generated abelian subgroup of finite index. □

Suppose X is a nonsingular projective algebraic surface over an algebraically closed field $\Bbbk$. It is called a *minimal model* if any birational morphism $X \to X'$ is an isomorphism. The theory of minimal models tells us that a minimal model satisfies one of the following properties

(i) K_X is *nef*, i.e. $K_X \cdot [C] \geq 0$ for any divisor class $[C]$ of a curve C on X,

(ii) there exists a morphism $X \to C$ to a nonsingular curve C with fibers isomorphic to $\mathbb{P}^1$;

(iii) $X \cong \mathbb{P}^2$.

For any X there exists a birational morphism $X \to Y$, where Y is a minimal model. In case (i), Y is unique (up to isomorphism) and is denoted by $X^{\min}$. Surfaces X admitting a unique minimal model $X^{\min}$ satisfying (i) are called surfaces of *non-negative Kodaira dimension*. We will explain the reason for the name a little later.

It is known that any birational morphism $f : X \to Y$ of nonsingular projective algebraic surfaces is equal to the composition of the birational morphisms

$$X = X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = Y,$$

where each morphism $f_i : X_i \to X_{i-1}$ is the *blow-up* of a point $x_i \in X_{i-1}$. This means that $E_i = f_i^{-1}(x_i) \cong \mathbb{P}^1$ and $f_i : X_i \setminus E_i \to X_{i-1} \setminus \{x_i\}$ is an isomorphism. The curve E_i on X_i is called an *exceptional curve* of the blow-up. Its divisor class $[E_i]$ satisfies $[E_i] \cdot [E_i] = -1$. Conversely, for any curve E on X with these properties there exists a blow-up $X \to X'$ with the exceptional curve E. We say also that the morphism is the *blowing down* of E. It follows that a minimal model is characterized by the property that it does not contain curves isomorphic to $\mathbb{P}^1$ with self-intersection equal to -1 (called (-1)-*curves*).

The images $y_1, \ldots, y_k$ of the points $x_1, \ldots, x_k$ in $\mathbb{P}^2$ under the composition of the blow-up maps could be a set of $< k$ points in $\mathbb{P}^2$. Netherless, abusing the definition, we say that X is isomorphic to the blow-up of points $x_1, \ldots, x_k$ in Y. Points x_j which are mapped to same point $y_i \in \mathbb{P}^2$ are called *infinitely near* to y_i. If all y_i are different, we will identify them with the points x_i (this happens then the composition $X_i \to Y$ is an isomorphism in an open neighborhood of x_i).

The other important corollary of the theory of minimal models is the isomorphism

$$\mathrm{Bir}(X^{\min}) \cong \mathrm{Aut}(X^{\min}),$$

where $\mathrm{Bir}(X)$ denotes the *group of birational isomorphisms* of X, isomorphic to the group of automorphisms of the field of rational functions on X acting identically on constants.

For any surface of non-negative Kodaira dimension, we have

$$\mathrm{Aut}(X) \subset \mathrm{Bir}(X) \cong \mathrm{Bir}(X^{\min}) \cong \mathrm{Aut}(X^{\min}).$$

All algebraic surfaces are divided into the four classes according to their *Kodaira dimension* $\mathrm{kod}(X)$ taking values in the set $\{-\infty, 0, 1, 2\}$. Recall that a divisor class D on an algebraic variety X is called *effective* if it is linearly equivalent to a non-negative linear combination of irreducible subvarieties of codimension 1. The set of all effective divisors in the same linear equivalence class of a divisor D is denoted by $|D|$. It is called the *complete linear system* associated to the divisor class $[D]$ of D. It can be identified with the projective space of lines $|D|$ in the linear space $H^0(X, \mathcal{O}_X(D))$, where $\mathcal{O}_X(D)$ is the invertible sheaf associated to the divisor D. If $|D| \neq \emptyset$, a choice of a basis $s_0, \ldots, s_N$ of $H^0(X, \mathcal{O}_X(D))$ defines a rational map $X \dashrightarrow \mathbb{P}^N$. It assigns to a point $x \in X$, the point in $\mathbb{P}^N$ with projective coordinates $[s_0(x), \ldots, s_N(x)]$. The pre-image of a hyperplane in $\mathbb{P}^N$ is a divisor from $|D|$. Every rational map $f : X \dashrightarrow \mathbb{P}^N$ is defined by a subspace of a complete linear system $|D|$.

The Kodaira dimension is defined by any of the following equivalent properties

- $\dim |mK_X|$ grows like $m^{\mathrm{kod}(X)}$ ($\mathrm{kod}(X) := -\infty$ if $|mK_X| = \emptyset$ for all $m > 0$);
- $\mathrm{kod}(X)$ is the dimension of the image of a rational map defined by $|mK_X|$ for $m >> 0$;
- $\mathrm{kod}(X)$ is the transcendence degree of the field of homogeneous fractions of the graded ring

$$R(X) := \oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X)).$$

The Kodaira dimension is a birational invariant of X.

Theorem 4.2. *Let X be a nonsingular projective surface over $\Bbbk$. Then $\mathrm{kod}(X) = -\infty$, if and only if X is birationally isomorphic to a minimal surface of type (ii) or (iii). If $\mathrm{kod}(X) \geq 0$, then one of the following cases occurs:*

(i) $\mathrm{kod}(X) = 2, K^2_{X^{\min}} > 0$ *;*

(ii) $\mathrm{kod}(X) = 1, K^2_{X^{\min}} = 0, k_{X^{\min}} \neq 0$*;*

(iii) $\mathrm{kod}(X) = 0, k_{X^{\min}} = 0$.

Corollary 4.3. *Suppose $\mathrm{Aut}(X)^*$ is an infinite group. Then one of the following cases occurs:*

(i) $\mathrm{kod}(X) = 0$ *or* 1*;*

(ii) there exists a birational morphism $f : X \to \mathbb{P}^2$.

Proof. It follows from Theorem 4.1 that $\mathrm{Aut}(X)^*$ is finite if $\mathrm{kod}(X) = 2$. Suppose $\mathrm{kod}(X) = -\infty$, and let $f : X \to Y$ be a birational morphism onto a minimal model different from $\mathbb{P}^2$. Let $\pi : Y \to C$ be a $\mathbb{P}^1$-fibration on Y. Since C is a curve, one can show that such a fibration is the projective bundle associated with a rank 2 vector bundle over C.

If $C \neq \mathbb{P}^1$, then the $\mathbb{P}^1$-fibration $\pi : Y \to C$ is unique. In fact, any other such fibration $\pi' : Y \to C'$ defines a surjective map from a general fiber of π' isomorphic to $\mathbb{P}^1$ to the curve C. This implies that C is a curve of genus 0 and hence isomorphic to $\mathbb{P}^1$. This show that π' does not exist. Now we have a natural homomorphism $\mathrm{Aut}(Y)^0 \to \mathrm{Aut}(C)^0$ whose kernel K is a group of automorphisms of Y that send any fiber of $\pi \circ f : X \to C$ to itself. Over an open subset of C, a fiber of $\pi \circ f$ is isomorphic to $\mathbb{P}^1$. A subgroup of the group of automorphisms of a projective algebraic variety that leaves invariant a closed algebraic subvariety is given by algebraic equations. Hence K is an algebraic subgroup of $\mathrm{Aut}(\mathbb{P}^1)$, and hence its group of connected components is finite. This shows that $\mathrm{Aut}_c(X)$ is finite.

Assume that $C \cong \mathbb{P}^1$. It follows from the classification of vector bundles on $\mathbb{P}^1$ that Y is isomorphic a *Segre-Hirzebruch surfaces* $\mathbf{F}_n, n \geq 0, n \neq 1$. If $n = 0$, the surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$

which is embedded in $\mathbb{P}^3$ via the Segre map. If $n \geq 2$, then $\mathbf{F}_n$ contains a unique section S_0 with $[S_0]^2 = -n$ such that the complete linear system $|nF + S_0|$, where F_0 is any fiber of π, maps Y to $\mathbb{P}^{n+1}$ with the projective cone over a rational normal curve R_n (the image of a Veronese map $\mathbb{P}^1 \to \mathbb{P}^n$). The pre-image of the vertex of the cone is the curve S_0 and the map is an isomorphism outside S_0 onto its image equal to the complement of the vertex. There is one possible case for a $\mathbb{P}^1$-fibration $X \to \mathbb{P}^1$ when X is isomorphic to the blow-up one point in $\mathbb{P}^2$. The fibration is obtained from the projection map from $\mathbb{P}^2$ to $\mathbb{P}^1$. The surface obtained in this way is denoted by $\mathbf{F}_1$, it is not a minimal model.

Let n be the smallest possible positive integer such that there exists a birational morphism $f : X \to Y \cong \mathbf{F}_n$. Let $f = f' \circ f_1 : X \to X_1 \to Y$, where $f_1 : X_1 \to Y$ is the blow-up of a point $x \in Y$. Let F be the fiber of $Y \to \mathbb{P}^1$ that contains x. Then its pre-image of F in X_1 is equal to the union of a curve $\bar{F}$ and the exceptional curve E. The usual properties of intersection theory tells us that $0 = F^2 = (\bar{F} + E)^2 = \bar{F}^2 + 2\bar{F} \cdot E + E^2 = \bar{F}^2 + 1$. Obviously $\bar{F}$ is isomorphic to F. Thus $\bar{F}$ is a (-1)-curve on X_1 which we can blow-down to get a birational morphism $f' : X_1 \to Y'$.

Suppose that $x \notin S_0$. Let $\bar{S}_0$ be the image of S_0 on Y'. Then $f'^{-1}(\bar{S}_0) = \bar{F} + S_0$, and using the intersection theory, we obtain as above, that $\bar{S}_0^2 = -n + 1$. Thus $Y' \cong \mathbf{F}_{n-1}$. If $n = 0$, we get $Y' \to \mathbf{F}_1$, and then composing $X \to X_1 \to \mathbf{F}_1 \to \mathbb{P}^2$, we obtain a birational morphism to $\mathbb{P}^2$.

So, we may assume that $x \in S_0$. Then a similar argument using the intersection theory gives that X_1 admits a morphism $X_1 \to Y' \cong \mathbf{F}_{n+1}$. If $n = 0$, we are done. If $n \geq 2$, then we obtain that Y' contains the section S_0 with $S_0^2 = -n - 1 \leq -3$. It is known that any curve on $\mathbf{F}_n$ different from the exceptional section S_0 has a non-negative self-intersection. This implies that the *proper-inverse transform* (i.e. the full pre-image minus the curves whose image under the birational map is a point) of S_0 on X is a unique curve S with self-intersection ≤ -3.[1] The group of automorphisms of X must leave S invariant. This gives a natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(S) \cong \mathrm{Aut}(\mathbb{P}^1)$ whose kernel and image is an algebraic group. This proves that $\mathrm{Aut}_c(X)$ is finite.

□

A rational surface X that admits a birational morphism to the projective plane will be called a *basic rational surface*.

If $\Bbbk = \mathbb{C}$, X is a smooth $2d$-manifold, so we can define the usual topological invariants of X such as the Betti numbers $b_i(X)$ or the Euler-Poincaré characteristic $e(X) = \sum(-1)^i b_i(X)$. In the general case, this can be also done by using the étale l-adic cohomology.

Let $\pi : X' \to X$ be the blow-up of a point $x \in X$ with the exceptional curve E. Then it is easy to see that $H^i(X;,\mathbb{Z}) = \pi^*(H^i(X,\mathbb{Z}))$ if $i \neq 2$, and

$$H^2(X,\mathbb{Z}) = \pi^*(H^2(X,\mathbb{Z}) \oplus \mathbb{Z}[E]. \tag{4.1}$$

[1] Here we use that the self-intersection of the proper transform of a curve under a blow-up a point decreases by 1.

In particular, we have

$$b_i(X') = b_i(X),\ i \neq 2, \quad b_2(X') = b_2(X) + 1,$$

and

$$\mathrm{Pic}(X') = \pi^*(\mathrm{Pic}(X)) \oplus \mathbb{Z}[E]. \tag{4.2}$$

The latter two equalities make sense over any field.

Another useful fact is that

$$K_{X'} = \pi^*(K_X) + [E]. \tag{4.3}$$

In particular,

$$K_{X'}^2 = K_X^2 - 1.$$

Applying Theorem 4.1, we obtain

Corollary 4.4. *Suppose* $\mathrm{Aut}(X)^*$ *is a non-elementary discrete group in* $\mathrm{O}(N^m(X)_{\mathbb{R}})$. *Then* $\mathrm{kod}(X) = 0$ *or* X *is the blow-up of* $N \geq 10$ *points in* $\mathbb{P}^2$.

The topological invariants are related to the algebraic invariants

$$q = \dim H^1(X, \mathcal{O}_X),\ p_g = \dim H^2(X, \mathcal{O}_X)$$

via the *Noether Formula*

$$12\chi(\mathcal{O}_X) = K_X^2 + e(X), \tag{4.4}$$

where $\chi(\mathcal{O}_X) = 1 - q + p_g$.

It is known that surfaces with Kodaira dimension 1 are elliptic sirfaces, i.e.they admit a morphism to a curve C such that the general fiber is a curve F with $K_F = 0$. Over $\mathbb{C}$ (or over any field of characteristic $\neq 2, 3$) it is a smooth curve of genus 1. The fibration is unique, and the group of automorphisms fits into an extension

$$1 \to K \to \mathrm{Aut}(X) \to G \to 1,$$

where G is a subgroup of $\mathrm{Aut}(C)$. The group K is a finitely generated abelian group. So, it follows that $\mathrm{Aut}_c(X)$ is an elementary group.

Theorem 4.5. *Let* X *be a minimal surface with* $\mathrm{kod}(X) = 0$. *Then* $K_X^2 = 0$ *and possible values of* $b_1(X), b_2(X), e(X), \chi(\mathcal{O}_X)$ *are given in the following table.*

Over $\mathbb{C}$, all K3 (resp. Enriques, resp. abelian) surfaces are diffeomorphic. A K3 surface is simply-connected, and the universal cover of an Enriques surface is of degree 2 and is a K3 surface. An abelian surface is a complex torus $\mathbb{C}^2/\Lambda$. We have

$$q = h^{1,0}(X) = \tfrac{1}{2} b_1(X)$$

Name	b_2	b_1	e	χ	K_X
K3	22	0	24	2	$K_X = 0$
Enriques	10	0	12	1	$2K_X = 0$
Abelian	6	4	0	0	$K_X = 0$
Bielliptic	2	2	0	0	$mK_X = 0, m = 2, 3, 4, 6$

Table 4.1: Surfaces with $\mathrm{kod}(X) = 0$

and

$$p_g = \dim H^0(X, \Omega_X^2) = \dim H^0(X, \mathcal{O}_X(K_X)).$$

Thus we have $q = 0, p_g = 2$ for a K3 surface, $q = 0, p_g = 0$ for an Enriques surface, and $q = 2, p_g = 1$ for an abelian surface. We will discuss these surfaces later.

We will not be interested in bielliptic surfaces. In this case

$$\mathrm{O}(N^1(X)_{\mathbb{R}}) \cong \mathrm{O}(\mathbb{R}^{1,1} \cong \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z},$$

so its discrete subgroups are not very interesting.

It is known that $N^1(X) = \mathrm{Pic}(X) = \mathrm{NS}(X)$ if X is a K3 surface. Also $\mathrm{Pic}(X) = \mathrm{NS}(X)$ and $N^1(X) = \mathrm{NS}(X)/\mathbb{Z}K_X$ for an Enriques surface.

It follows from (1.6) that the Picard number of a complex surface of Kodaira dimension 0 satisfies the following inequality

$$\begin{cases} 1 \leq \rho \leq 20 & \text{if } X \text{ is a K3 surface,} \\ \rho = 10 & \text{if } X \text{ is an Enriques surface,} \\ 1 \leq \rho \leq 4 & \text{if X abelian surface,} \\ 2 & \text{if X is a bielliptic surface.} \end{cases}$$

Note that in positive characteristic it is possible that the Picard number of a K3 surface (resp. abelian surface) takes the value 22 (resp. 6) but not 21 (resp. 5).

Let X be a K3 surface over $\mathbb{C}$. Then $H^2(X, \mathbb{Z})$ has no torsion, so the Poincaré Duality Theorem implies that $H^2(X, \mathbb{Z})$ is a unimodular lattice of rank 22. The Hodge Index Theorem gives that the signature of $H^2(X, \mathbb{R})$ is equal to $(3, 19)$. Using the fact that $K_X = 0$, one can also prove that $H^2(X, \mathbb{Z})$ is an *even lattice*.[2] Applying *Milnor's Theorem* about classification of *unimodular* indefinite quadratic lattices (see [48]), we obtain that

$$H^2(X, \mathbb{Z}) \cong \mathsf{U} \perp \mathsf{U} \perp \mathsf{U} \perp \mathsf{E}_8 \perp \mathsf{E}_8. \tag{4.5}$$

[2] A quadratic lattice is called even if its quadratic form takes only even values and is called unimodular if the determinant of any symmetric matrix representing it is equal to ± 1.

Here and later we denote by U (resp. E_8) the unique even unimodular even quadratic lattice of signature $(1,1)$ (resp. unique unimodular even negative definite lattice of rank 8). The numerical lattice $N^1(X)$ is a primitive sublattice of this lattice.

Let X be an Enriques surface over $\mathbb{C}$. Then K_X generates the torsion subgroup of $H^2(X,\mathbb{Z})$ and

$$N^1(X) = H^2(X,\mathbb{Z})/\mathbb{Z}K_X.$$

Again, by the Poincaré Duality, we obtain that the lattice $N^1(X)$ is a unimodular lattice of rank 10 and signature $(1,9)$. Since $k_X = 0$, the Riemann-Roch Theorem implies that the lattice is even. Applying Milnor's Theorem, we obtain that

$$N^1(X) \cong \mathsf{U} \perp \mathsf{E}_8. \tag{4.6}$$

Let X be an abelian surface. Similar arguments give that

$$H^2(A,\mathbb{Z}) \cong \mathsf{U} \perp \mathsf{U} \perp \mathsf{U}$$

and $N^1(X)$ is a primitive sublattice of $H^2(A,\mathbb{Z})$ of signature $(1,\rho-1)$.

Example 4.6. Assume X is a complete intersection in $\mathbb{P}^{n+d}$ of dimension $n = 2k$ with $k(k-1)/2$ even. Let $(a_1,\dots,a_d)$ be the degrees of the hypersurfaces that cut out X. Then

$$K_X = (n+d+1-a_1-\cdots-a_d)h,$$

hence $K_X^k = (n+d+1-a_1-\cdots-a_d)^k h^k$. This implies that, if $n+d+1-a_1-\cdots-a_d > 0$, $N^1(X)_0$ is negative definite, and G is finite. If $K_X \neq 0$, the group G of connected components of $\mathrm{Aut}(X)$ is finite. The only interesting case is when X is a Calabi-Yau, i.e.

$$n+d+1-a_1-\cdots-a_d = 0.$$

For example, X is a quartic surface, or X is a 6-dimensional Calabi-Yau.

Lecture 5

Reflection groups of isometries

It is known that the orthogonal group $\mathrm{O}(1,n)$ of the space $\mathbb{R}^{1,n}$ is generated by *reflections*

$$s_e : v \mapsto v - 2\frac{(v,e)}{(e,e)}e,$$

where $(e,e) \neq 0$. If $(e,e) > 0$, then s_e has only one fixed point in $\overline{\mathbb{H}}^n$, the point $[e] \in \mathbb{H}^n$. On the other hand, if $(e,e) < 0$, then the set of fixed points is a hyperplane $H_e(0)$ in $\mathbb{H}^n$. If we write $H_e(0)$ by the equation $a_0x_0 - \sum a_ix_i = 0$, then its pre-image in the Poincaré model is given by the equation $2(\sum a_iu_i) = a_0(1+|u|^2)$. We can rewrite the equation in the form $|u-c|^2 = -1 + \sum \frac{a_i^2}{e_0^2}$, where $c = (\frac{a_1}{e_0}, \dots, \frac{a_1}{a_0})$. It is a sphere with center c and radius-square greater than 1. The reflection transformation is the inversion with respect to this sphere.

From now on, we will consider only reflections s_e with $(e,e) < 0$. It is obvious that $s_{ke} = s_e$ for any scalar $k \neq 0$, so we may assume that $(e,e) = -1$. The composition of two reflections s_e and $s_{e'}$ depends on (e,e'). If $(e,e') = 0$, they commute, so that $s_e \circ s_{e'}$ is an involution. If $(e,e') < 1$, then the plane spanned by e, e' is negative definite, so $s_e \cdot s_{e'}$ is of finite order. It is a rotation in the dihedral angle $2\arccos(e,e')$ with the edge defines by the intersection of the hyperplanes $H_e(0) \cap H_{e'}(0)$. Here is the picture in $\mathbb{H}^2$.

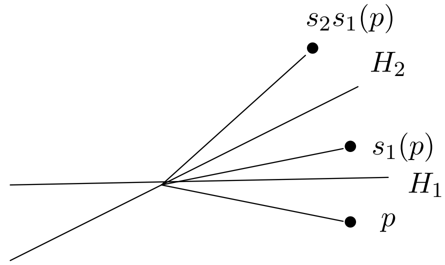


Figure 5.1: Product of two reflections

If $(e, e') > 1$, then it is a hyperbolic transformation.

Let $e_1, \ldots, e_N$ be N vectors with $(e_i, e_i) = -1$. Let Γ be the group of isometries of $\mathbb{H}^n$ generated by the reflections $s_1 = s_{e_1}, \ldots, s_n = s_{e_n}$. If Γ is a discrete group, then the vectors e_i must satisfy the following properties:

- $(e_i, e_i) > 1$ or $(e_i, e_i) = \cos \frac{\pi}{m_{ij}}$, where $m_{ij} \in \mathbb{Z} \cup \{\infty\}$.

Such a group is called a *hyperbolic Coxeter group*. The convex polyhedron

$$P_\Gamma = \bigcap_{i=1}^{N} H_{e_i}^-, \tag{5.1}$$

is a fundamental domain of Γ in $\mathbb{H}^n$. A convex polyhedron in $\mathbb{H}^n$ of this form is called a *Coxeter polyhedron*.

The group (Γ, S) together with its set of generators $S = \{s_1, \ldots, s_N\}$ is an abstract Coxeter group. This means that a set of generators S is chosen in Γ such that Γ can be defined the relations

$$(s_i s_j)^{m_{ij}} = 1, m_{ii} := 1.$$

The matrix $C(\Gamma) = ((m_{ij}))$ is called the *Coxeter matrix* of the Coxeter group Γ. All Coxeter matrices (and hence Coxeter groups) of finite Coxeter groups are classified. They are called of *finite type* (or *spherical type*). The groups can be considered as reflection group acting in the *spherical geometry* S^n of constant sectional curvature 1. The reflection groups fixing a point on the absolute can be considered as reflection groups in the Euclidean space E^n. They are also classified.

The classification of an abstract Coxeter groups $\Gamma, S)$ is given in terms of its *Coxeter diagram*. Its vertices v_s correspond to the generators $s \in S$. Two vertices $v_s i, v_{s'}$ are joined by $m_{ss'} - 2$ edges or by an edge with mark $m_{ss'}$ if $m_{ss'} < \infty$. If $m_{ss'} = \infty$, the vertices are joined by a thick edge or by an edge with mark ∞. In the case when Coxeter groups comes as a reflection group defined by a Coxeter polyhedron, one distinguishes thick edges by using the dotted edge if $(e_i, e_j) > 1$ and the thick edge if $(e_i, e_j) = 1$.

One extends the previous notions to the case when the set of vectors $e_1, \ldots, e_N$ is an infinite countable set. This is often used in algebraic geometry.

For any discrete group Γ of isometries of $\mathbb{H}^n$ let Γ_r denotes the subgroup generated by all reflections contained in Γ. Then Γ_r is a normal subgroup of Γ and

$$\Gamma = \Gamma_r \rtimes A(P),$$

where $A(P)$ is the intersection of γ with the group of symmetries of the Coxeter polyhedron of Γ_r.

Let $M \subset R^{1,n}$ be a lattice in $\mathbb{R}^{1,n}$. A reflection s_e preserves M if and only if, for all $v \in M$,

$$2\frac{(v, e)}{(e, e)} e \in M.$$

Replacing e by some scalar multiple, we may assume that $e \in M$. We also must have that, for any$v \in M$,

$$\frac{2(v,e)}{(e,e)} \in \mathbb{Z}. \tag{5.2}$$

Equivalently, this means that the vector $e^* = \frac{2}{(e,e)}e \in M^\vee := \mathrm{Hom}(M,\mathbb{Z})$, where we identify any vector x in $\mathbb{R}^{1,n}$ with the linear function $y \mapsto (x,y)$. In particular, any reflection s_e with $e \in M$ and $(e,e) = -2$ or $(e,e) = -1$, defines a reflection transformation of M. We call elements $\alpha \in M$ satisfying (5.2) *roots*. We will also assume a root is a *primitive vector*, i.e. $M/\mathbb{Z}\alpha$ has no torsion.

A Coxeter polytope P in $\mathbb{H}^n$ is called a *lattice polytope* if the reflection group Γ_P preserves some lattice M in $\mathbb{R}^{1,n}$.

Proposition 5.1. *A Coxeter polytope P with the Gram matrix $G(P) = (g_{ij})$ is a lattice polytope if and only if there exist some real numbers $c_1,\ldots,c_N$ such that*

$$c_ic_jg_{ij} \in \mathbb{Z}. \tag{5.3}$$

Proof. Suppose Γ_P preserves a lattice M. By above, there exist some real numbers $c_1,\ldots,c_N$ such that $c_ie_i \in M$ and, for all $v \in M$,

$$\frac{2(v,c_ie_i)}{(c_ie_i,c_ie_i)} = \frac{2(v,e_i)}{c_ie_i} \in \mathbb{Z}.$$

Substituting $v = c_je_j$, we obtain that $\frac{c_ig_{ij}}{c_j} \in \mathbb{Z}$. Also, we have $(c_ie_i,c_ie_i) = -c_i^2 \in \mathbb{Z}$. Hence, $c_ic_jg_{ij} \in \mathbb{Z}$. Conversely, assume that the conditions are satisfied. Then the subgroup M of $\mathbb{R}^{1,n}$ generated by $v_i = c_ie_i$ is a Γ_P-invariant quadratic lattice with $a_{ij} = (v_i,v_j) = c_ic_j(e_i,e_j) \in \mathbb{Z}$.

We say that a vector $(c_1,\ldots,c_{n+1})$ satisfying (5.3) is a *multiplier vector* of a lattice polytope. We always assume that the multiplier is chosen with minimal possible product $c_1\cdots c_{n+1}$.

□

Example 5.2. Suppose the Coxeter diagram contains an edge with mark $m > 3, m \neq 4,6$. Then the Coxeter polytope is not a lattice polytope. In fact $\cos\pi/m$ does not belong to any quadratic extension of $\mathbb{Q}$ and hence cannot be expressed as c_ic_jn, where c_i^2, c_j^2, n are integers.

Let us denote by $\iota : M \to M^\vee$, the natural homomorphism obtained by the restriction of the function $(v,)$ to M. The quotient group $A_M = M^\vee/\iota(M)$ is a finite abelian group isomorphic to the group defined by an integral matrix of the symmetric bilinear form (x,y) on M. It is called the *discriminant group* of M. It is equipped with a quadratic map

$$q_{A_M} : A_M \to \mathbb{Q}/\mathbb{Z},\ q_{A_M}(x+M) = x^2 \mod \mathbb{Z},$$

where we consider $M^\vee$ as a subgroup of $M \otimes \mathbb{R} = \mathbb{R}^{1,n}$. This quadratic form plays an important role in the classification of quadratic lattices (see [41]).

Let $\mathrm{Ref}(M)$ denote the subgroup of the orthogonal group $\mathrm{O}(M)$ of M generated by reflections lattice M in $\mathbb{R}^{1,n}$. It is called the *reflection subgroup* of M. We will be mostly dealing with even lattices. For each lattice M one can consider the largest even sublattice M^{ev}. Since $(x+y,x+y) = (x,x)+(y,y)+2(x,y)$, we see that M^{ev} is generated by all vectors with even norm. It is clear that any isometry of M leaves M^{ev} invariant, so we have an inclusion of the groups

$$\mathrm{O}(M) \subset \mathrm{O}(M^{\mathrm{ev}}).$$

Let $\mathrm{O}(M^{\mathrm{ev}}) \to \mathrm{O}(A_{M^{\mathrm{ev}}}, q_{A_{M^{\mathrm{ev}}}})$ be the natural homomorphism. Then an isometry of M^{ev} lifts to an isometry of M if and only it belongs to the kernel of this homomorphism. Thus, the index of $\mathrm{O}(M^{\mathrm{ev}})$ in $\mathrm{O}(M)$ is finite.

The relationship between the reflection groups of M and M^{ev} is rather complicated. A root of M^{ev} is not necessary a root of M, a root of M with even norm is a root of M^{ev}. However, if α is a root of M with odd norm, then 2α is not necessary a root of M^{ev}. The exception is when the norm of α is equal to -1. Then 2α is a root of M^{ev} of norm -4.

For any positive integer k, let $\mathrm{Ref}_k(M)$ be the subgroup of $\mathrm{Ref}(M)$ generated by reflections in roots of norm $-k$. Obviously, each subgroup $\mathrm{Ref}_k(M)$ is a normal subgroup of $\mathrm{Ref}(M)$. We say M is a *reflective lattice* (resp. k-reflective lattice) if the subgroup $\mathrm{Ref}(M)$ (resp. $\mathrm{Ref}_k(M)$) is of finite index in $\mathrm{O}(M)$.

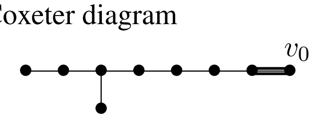
The following is a very useful fact (see [53]).

Proposition 5.3. *Suppose a system of generators S of a Coxeter group Γ is divided into two disjoint subsets S_1 and S_2 such that the order of any $s_1s_2, s_1 \in S_1, s_2 \in S_2$, is either even or infinite. Then*

$$\Gamma = N \rtimes \Gamma_1,$$

where Γ_1 is generated by S_1 and N is the minimal normal subgroup containing S_2.

Example 5.4. Consider the following Coxeter diagram



Let S_1 be the vertex v_0. Then all other vertices define reflections s_i that either commute with the reflection s_0 defined by v_0 or s_is_0 is of infinite order. This shows that s_0 is not conjugate to any s_i. Since the group generated by $s_i, i \neq 0$ is isomorphic to the Weyl group of root system of type E_8, we obtain that

$$\Gamma = W(E_8) \rtimes \langle\langle s_0 \rangle\rangle.$$

This implies, for example. that any two se groups are related

Theorem 5.5. *Let M be a k-reflective lattice and P be the Coxeter polytope of the reflection group $\Gamma = \mathrm{Ref}_k(M)$. Then it is spanned by a finite set of points $[v_1], \ldots, [v_N]$ in $\overline{\mathbb{H}}^n$. The faces H_e*

correspond to normalized roots of norm $-k$. Each parabolic subdiagram of the Coxeter diagram is a connected component of a parabolic subdiagram with number of vertices equal to $n+1$. Conversely, each such diagram is the Coxeter diagram of of the Coxeter group $\mathrm{Ref}_k(M)$ of finite index in $\mathrm{O}(M)$.

Remark 5.6. The condition that Γ is of finite index in $\mathrm{O}(M)$ is equivalent to that $\mathbb{H}^n/\Gamma$ is of finite volume (in the group theory this is expressed by saying that Γ is a lattice in the Lie group $\mathrm{O}(1,n)'$). This is equivalent to that the fundamental polyhedron of Γ is bounded, i.e. it spanned by a finite set of vertices lying in the closure of $\mathbb{H}^n$. If none of these points lie on the absolute, the orbit space $\mathbb{H}^n/\Gamma$ is compact (this is expressed by saying that Γ is *cocompact* or a *uniform lattice* in $\mathrm{O}(1,n)'$. Hyperbolic Coxeter groups which are simplices in $\mathbb{R}^{1,n}$ (i.e. generated by $n+1$ vertices) are called *quasi-Lanner groups* (*Lanner groups* if they are cocompact). All such diagrams, and hence groups are classified. Of course, not all of them are realized in reflective lattices.

Example 5.7. Let $I^{1,n}$ denote the odd (i.e. not even) unimodular quadratic lattice of signature $(1,n)$. By Milnor's Theorem, it is isomorphic to the sublattice of $\mathbb{R}^{1,n}$ of vectors with integer coordinates. The Table 5.1 lists the Coxeter diagrams of reflection groups of odd unimodular lattice $I^{1,n} = \mathbb{Z}^{1,n} \subset \mathbb{R}^{1,n}$ for $n \le 17$.

The lattices $I^{1,18}$ and $I^{1,19}$ are also reflective. They contains 37 (resp. 50) vertices (see [53]). Here all roots are of norm -2 except those which are joined by a thick edge. They are of norm -1. It follows from the previous discussion that M^{ev} is reflective if M is reflective and all its roots have even norm or of norm -1.

Example 5.8. Let $\langle m\rangle$ denote a lattice of rank 1 generated by an element with the norm indicated inside the bracket. For example, the unimodular odd lattice $I^{1,n}$ considered in the previous example is isomorphic to the orthogonal sum of the lattice $\langle 1\rangle$ and n copies of the lattice $\langle -1\rangle$. Consider the free abelian group H generated by $h_1,\ldots,h_{p-1},e_1,\ldots,q+r$. Let $H^\vee = \mathrm{Hom}(H,\mathbb{Z})$ be the dual group and $h^1,\ldots,h^{p-1},-e^1,\ldots,-e^{q+r}$ be the dual basis. Let

$$\alpha_0 = h_1 - e_1 - \cdots - e_q,\ \alpha_1 = h_1 - h_2,\ \alpha_{p-2} = h_{p-2} - h_{p-1},$$

$$\alpha_{p-1} = e_1 - e_2,\ \alpha_{q+r-1} = e_{q+r-1} - e_{q+r}$$

be vectors from H. Let

$$\alpha^0 = (q-2)h^1 + (q-1)h^2 + \cdots + (q-1)h^{p-1} + e^1 + \cdots + e^q,$$

$$\alpha^1 = -h^1 + h^2, \ldots, \alpha^{p-2} = -h^{p-1} + h^p, \alpha^{p-1} = -e_1 + e_1, \ldots,\ \alpha_{q+r-1} = -e_{q+r-1} + e^{q+r}.$$

Let $\mathsf{E}_{p,q,r}$ denote the free abelian group generated by α_0, α_{q+r-1} equipped with a symmetric bilinear form defined on the basis by

$$(\alpha_i,\alpha_j) = \alpha^i(\alpha_j).$$

The following graph of type $T_{p,q,r}$ is the incidence graph of the matrix $I_{q+r} + ((\alpha_i,\alpha_j))$. In other words, the diagram means that each α_i has the norm equal to -2, $(\alpha_i,\alpha_j) = 1$ if the corresponding vertices are joined by an edge, and $(\alpha_i,\alpha_j) = 0$ otherwise. The diagram is the Coxeter diagram of

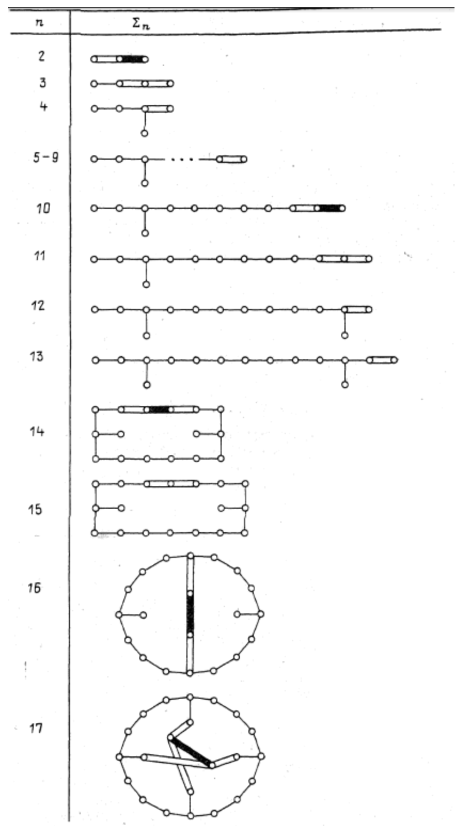


Table 5.1: Reflection groups of lattices $I^{1,n}$

a reflection group (denoted by $W_{p,q,r}$) in the inner product space $\mathsf{E}_{p,q,r} \otimes \mathbb{R} \cong \mathbb{R}^{q+r}$. The signature of the quadratic form is equal to

$$\mathrm{sign}(\mathsf{E}_{p,q,r}) = \begin{cases} (0, q+r) & \text{if } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, \\ (1, q+r-1) & \text{if } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \end{cases}$$

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, the quadratic form has one-dimensional radical, the quadratic form in the quotient space by this radical is negative definite. The discriminant group of the lattices $\mathsf{E}_{\mathsf{p,q,r}}$ can be computed. Its order is equal to $pqr - pq - pr - qr$.

We assume that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Other cases lead to the reflection groups in the spherical spaces or in the Euclidean space. We assume that $p \leq q \leq r$. The finite reflection groups correspond to the cases $p = 1, q + r = n$, or $(p, q, r) = (2, 2, n), n \geq 2$, or $(p, q, r) = (2, 3, 3), (2, 3, 4), (2, 3, 5)$. They lattices in these cases

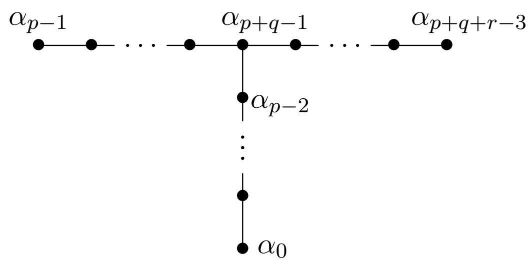


Figure 5.2: T_{pqr} graph

are called the root lattices and denoted by $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, respectively. The cases $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ occur if and only if $(p,q,r)=(3,3,3),(2,4,4),(2,3,6)$. The lattices are denoted by $\tilde{\mathsf{E}}_6, \tilde{\mathsf{E}}_7, \tilde{\mathsf{E}}_8$, respectively. They correspond to affine root systems of types E_6, E_7, E_8. The corresponding Coxeter groups fit in the exact sequence

$$0 \to \mathbb{Z}^r \to W(\tilde{\mathsf{E}}_{p,q,r}) \to W(\mathsf{E}_{\mathsf{p,q,r}}) \to 1.$$

So the groups, when realized as subgroups of $\mathrm{Iso}(\mathbb{H}^n)$ are elementary groups.

It is clear that $\mathrm{Ref}(\mathsf{E}_{p,q,r}) = \mathrm{Ref}_2(\mathsf{E}_{p,q,r})$. The following triples (p,q,r) correspond to reflective lattices of hyperbolic signature:

$$(2,3,n),\ 7 \le n \le 10,\ (2,4,5),\ (2,4,6),\ (3,3,4),(3,3,5),\ (3,3,6).$$

A set of roots in a lattice M is called a *crystallographic root basis* if the reflection group generated by the reflections in these roots is of finite index in $\mathrm{O}(M)$. The set of vectors $\alpha_0,\ldots,\alpha_{q+r}$ corresponding to the vertices of the graph $T_{p,q,r}$ is crystallographic only in the three cases

$$(p,q,r) = (2,3,7),\ (2,4,5),\ (3,3,4).$$

We will denote the lattice $\mathsf{E}_{2,3,N-3}$ by E_N. The lattices E_8 and E_{10} are distinguished from other lattices $\mathsf{E}_{\mathsf{P,q,r}}$ by the property that it is the only *unimodular lattice*, that its discriminant group is trivial.

Note that the three lattices from above contain other crystallographic bases. For example, the following Coxeter diagrams correspond to crystallographic bases in E_{10} of cardinality $k \le 12$.

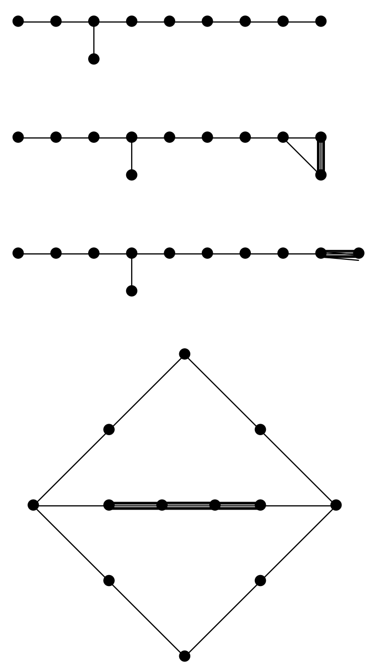


Figure 5.3: Crystallographic root basis in E_{10} of cardinality ≤ 12

Lecture 6

Reflection groups in algebraic geometry

We start with an example in which we realize the *universal Coxeter group* $\mathsf{UC}(N)$. Recall that a Coxeter group is given by its set of generators $s_1, \ldots, s_N$ and relations $(s_i s_j)^{m_{ij}} = 1$. It is the quotient group of the free product $\mathsf{UC}(N)$ of N copies of the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

First we have to remind the reader about the *blow-up* construction in algebraic geometry. In the simpelst case, the constructionn is a sort of surgery, it replaces a closed smooth subvariety Z of an smooth algebraic variety X with the projectivization E of its normal bundle. The result is a smooth algebraic variety $\mathrm{Bl}_Z(X)$ that comes with the natural regular map $\pi : \mathrm{Bl}_Z(X) \to X$ which is an isomorphism over $X \setminus Z$ and its fibers over points $z \in Z$ are projective spaces of dimension equal to $\mathrm{codim}_z(Z, X)$. The map π is called the *blowing-down map* (or *blowing-up* map). The subvariety E is a hypersurface in $\mathrm{Bl}_Z(X)$. It is called the *exceptional divisor*. For example, the rational projection map $p_L : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k-1}$ from a k-plane L can be extended to a regular map $\mathrm{Bl}_L(\mathbb{P}^n) \to \mathbb{P}^{n-k-1}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Bl}_L(\mathbb{P}^n) & & \\ {\scriptstyle \pi}\downarrow & \overset{\tilde{p}_L}{\searrow} & \\ \mathbb{P}^n & \overset{p_L}{\dashrightarrow} & \mathbb{P}^{n-k} \end{array} \tag{6.1}$$

In this case $\tilde{p}_L$ is a projective bundle, its fibers are projective spaces of dimension $k+1$ such that each fiber of $p_L : \mathbb{P}^n \setminus L \to \mathbb{P}^{n-k-1}$ is equal to the complements of a hyperplane in it. Each such hyperplane is mapped isomorphically to L under the map π. In general, the blow-up of a nonsingular variety along a nonsingular subvariety is a local analog of this example. One choose local coordinates equations $z_1 = \ldots = z_r$ of Z in some open affine neighborhood U of X and considers the subvariety of $U \times \mathbb{P}^{r-1}$ defined by the equations $t_i z_j - t_j z_i = 0, i \neq j$. The projection $(z, t) \mapsto z$ has fibers isomorphic to $\mathbb{P}^{r-1}$, if $z_i \neq 0$ in U, then we can express $t_i = z_j / z_i$ for all j. This shows that the projection is an isomorphism outside Z. This construction can be globalized by gluing together these local constructions.

One can extend this construction to not necessarily nonsingular algebraic varieties. Let $\pi : \mathrm{Bl}_Z(X) \to X$ be the blow-up of Z, and Y be a closed subvariety of X not contained in Z. We consider $V = \pi^{-1}(Y \setminus (Y \cap Z)) \subset \mathrm{Bl}_Z(X) - E \cong X \setminus Z$, and then take the closure of V in the Zariski topology. We call this closure the *proper inverse transform* of Y under the blow-up $\mathrm{Bl}_Z(X) \to X$ and denote it by $\pi^{-1}(Y)$, it should not be confused with the inverse-image of Y under π. The restriction of the map π is a regular map $\pi^{-1}(Y) \to Y$. It is an isomorphism over $Y \setminus (Y \cap Z)$ and the pre-image of $Y \cap Z$ is a closed subvariety of E. If we are lucky, repeating this construction we obtain a regular map $\tilde{Y} \to Y$ which is a *resolution of singularities*. This means that $\tilde{Y}$ is nonsingular and the map is an isomorphism over the open locus Y^{sm} of nonsingular points.

Example 6.1. Let Y be a quadric cone Q in $\mathbb{P}^3$. This means that Y is defined by a quadratic equation $q(t_0, t_1, t_2, t_3, t_4) = 0$, where q is a quadratic form with one-dimensional radical. After changing the coordinates, we may assume that

$$Q : t_0 t_1 - t_2^2 = 0.$$

We already know that the Segre-Hirzebruch surface $\mathbf{F}_2$ admits a birational morphism onto Y that blows down the exceptional section S_0 to the singular point of Y. We can see it in another way as follows. Consider the projection map $\mathrm{Bl}_{x_0}(\mathbb{P}^3) \to \mathbb{P}^2$ as in (6.1) from the singular point $x_0 = [0, 0, 0, 1]$ of Q. The points of the exceptional divisor $E \cong \mathbb{P}^2$ are viewed as the directions of lines passing through the point x_0. The lines that lie on Q, i.e. the lines with parameter equations $s[a_0, a_1, a_2, a_3] + t[0, 0, 0, 1]$, where $a_0 a_1 = a_2^2$, define the subvariety of E isomorphic to a nonsingular conic. The proper inverse transform $\tilde{Q} = \pi^{-1}(Q) \to Q$ is a resolution of singularities. It replaces x_0 with a conic. The restriction of the projection $\mathrm{Bl}_{x_0}(\mathbb{P}^3) \to \mathbb{P}^2$ to $\tilde{Q}$ is isomorphic to the $\mathbb{P}^1$-bundle over a conic in $\mathbb{P}^2$.

Now, we are in business. Consider a quartic surface X' in $\mathbb{P}^3$. By definition, it is given by equation

$$X' : F_4(t_0, t_1, t_2, t_3) = 0 \tag{6.2}$$

where F_4 is a homogeneous polynomial of degree 4. Assume that X' has an ordinary double point p_0. This means that, if we write X' in affine coordinates near the point p_0 as the surface $f(x, y, z) = 0$, then the Taylor expansion of f at p_0 starts with a non-degenerate quadratic form $f_2(x, y, z)$. Consider the map $X' \setminus \{p_0\} \to X'$ that assigns to a point x the unique point on the line joining x with p_0. Since x_0 is a double point, the polynomial F_4 restricted to the line has a double root at p_0, so there must be the fourth root x'. This point we assign to x. Of course, the point x could be also a double root on the line, hence x' could be equal to x. Also, p_0 could be a root of multiplicity 3, then $x' = p_0$. The map defines a birational automorphism g_{p_0} of order 2 of the surface X'. Let $\sigma : X \to X'$ be the proper transform of X' in the blow-up of $\pi : \tilde{\mathbb{P}}^3 \to \mathbb{P}^3$ with center at x_0. Since, locally near p_0, the surface is isomorphic to the quadratic cone from the previous example, we obtain X' Then the σ^{-1} is equal to the conic E in the plane $\pi^{-1}(p_0)$. Over a neighborhood of p_0, the surface X is now nonsingular. Now we can extends the g_{p_0} to X. Each point $x \in E$ corresponds to a line through p_0 that intersects X' with multiplicity 3 at this point.

We assign to x the fourth intersection point of the line with X'. This gives a construction of an automorphism g_{p_0} associated to a double points p_0.

One can do it in formulas. Without loss of generality, we may assume that p_0 has the coordinates $[1, 0, 0, 0]$, so that we can write the equation of X' in the form

$$t_0^2 Q(t_1, t_2, t_3) + 2t_0\Phi_3(t_1, t_2, t_3) + Phi_4(t_1, t_2, t_3) = 0. \tag{6.3}$$

The line passing through the point x_0 has a parameter equation $[s, ta_1, ta_2, ta_3]$. Plugging this in the equation and cancelling by t^2, we obtain

$$s^2 Q(a_1, a_2, a_3) + 2st\Phi_3(a_1, a_2, a_3) + t^2\Phi_4(a_1, a_2, a_3) = 0. \tag{6.4}$$

This shows that it intersects the surface at two points corresponding to parameters $[s, t]$. So, we choose one point corresponding to $[s_1, t_1]$, the second solution $[s_2, t_2]$ gives us the point x'. The line intersects X' at x_0 with multiplicity 3 if $[1, 0]$ is a solution of (6.4). This happens if $Q(a_1, a_2, a_3) = 0$.

We view the point $[a_1, a_2, a_3]$ as the slope of the space line passing through p_0. So, the slopes corresponding to the lines intersecting X' only at one point with multiplicity 1 besides p_0 is a conic $Q(z_1, z_2, z_3) = 0$. Or better, we may consider the projection of X to the plane with equation $t_0 = 0$ from the point p_0. Then the line corresponds to a point on the plane. So, the lines passing through p_0 with multiplicity 3 correspond to the points on the conic $Q(t_0, t_1, t_2) = 0$. This is the same conic that appears in the resolution of singularity p_0 defined by the projection from the point p_0. So, the direction $x = [a_1, a_2, a_3]$ should be considered as a point on the surface $\tilde{X}$. We extend our map $x \mapsto x'$ by assigning to the point x the second solution of (6.4) defined by $[s_2, t_2] = [-\Phi_4(a_1, a_2, a_3), \Phi_3(a_1, a_2, a_3)]$. The only case where this does not make sense is when $Q(a_1, a_2, a_3) = \Phi_4(a_1, a_2, a_3) = \Phi_3(a_1, a_2, a_3) = 0$. This happens if and only if the line is contained in the surface. One can show that the map $T : x \mapsto x'$ still extends to X and fixes any point on this line including the point (a_1, a_2, a_3) lying on the exceptional curve of $\sigma : X \to X'$. Other fixed points of the automorphism $g_{p_0} : X \to X$ lie on the line with the slope $[a_1, a_2, a_3]$ satisfying the discriminant equation of degree 6

$$D(t_1, t_2, t_3) = \Phi_3(t_1, t_2, t_3)^2 - Q(t_1, t_2, t_3)\Phi_4(t_1, t_2, t_3) = 0.$$

These line passes through p_0 and tangent to X' at some other point. Note that the conic $Q(t_1, t_2, t_3) = 0$ is tangent to the plane curve $D = 0$ at the points $\Phi_3 = Q = 0$. In general case, we have 6 points of the intersection.

Let us see how the automorphism g_{x_0} of X acts on the group $N^1(X)$. Let R be the exceptional curve over p_0, and $\tilde{H}$ be the pre-image of a plane section H of X' on X. If H passes through p_0, then it is projected to a line in the plane $t_0 = 0$. The images $g_{p_0}(x)$ of all its points except p_0 stay in the plane. The pre-image $\tilde{H}$ of this plane on X is the union of the proper transform $\bar{H}$ of H and the exceptional curve R. Let $h = [\tilde{H}]$, obviously, it does not depend on H, and $r = [R]$ be the cohomology class of R. We see that the class $h - r$ is invariant with respect to

$g_{p_0}^*$. Let l be the class of a line in $\mathbb{P}^2 : t_0 = 0$. The cohomology class of the pre-image of l under the projection is the class $h - r$. We know that $T(R)$ is the conic in the plane $t_0 = 0$. Thus $g_{p_0}^*(r) + r = 2(h - r)$, hence $g_{p_0}^*(r) = 2h - 3r$. We also know that $g_{p_0}^*(h - r) = h - r$. This gives $g_{p_0}^*(h) - h - r + g_{p_0}^*(r) = h - r + (2h - 3r) = 3h - 4r$. We have $g_{p_0}^*(h - r) = h - r$. Thus $g_{p_0}^*$ leaves the span $\mathbb{Z}h + \mathbb{Z}r$ invariant and acts on this sublattice as a matrix

$$A = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}.$$

Suppose that X' has other nodes $p_1, \ldots, p_{N-1}$. Let R_i be the classes of the exceptional curves obtained by projections from each point p_i. Obviously, we can find a plane H that misses all singular points. Thus $h \cdot [R_i] = 0$. Obviously $R_i \cap R = \emptyset$. Thus $[R_i]$ is orthogonal to h and r.

Consider the reflection s_α with respect to the class $\alpha = \frac{1}{2}(h - 2r) \in \mathrm{Pic}(X)_\mathbb{Q}$ of norm -1. Then, we immediately check that s_α acts on h, r via the matrix A. It also acts as the identity on the orthogonal complement of $h - 2r$, hence it fixes the classes $[R_i], i \neq 0$.

Suppose that $N^1(X) \otimes \mathbb{Q}$ is generated by the class of a plane section h and the classes r_i of the exceptional curves R_i. In other words, the rank of $N^1(X)$ is equal to $1 + N$, where N is the number of nodes on X. Then, each node defines an automorphism $T_i = g_{p_i}$ of order 2 of X that acts on $N^1(X)$ as the reflection s_{α_i}, where $\alpha_i = h - 2r_i$ of norm -4. We have $(\alpha_i, \alpha_j) = (h - 2r_i, h - 2r_j) = 4$. After normalizing the vectors to get the vectors $e_i = \frac{1}{2}\alpha_i$ of norm -1, the Gram matrix of the vectors $e_1, \ldots, e_N$ becomes the circulant matrix $\mathrm{circ}(-1, 1, \ldots, 1)$. Its Coxeter diagram is the complete graph with n vertices and thick edges. Since all $m_{ij} = \infty$, we obtain that there are no relations except the relation $s_{e_i}^2 = 1$. We get the universal Coxeter group $\mathsf{UC}(N)$ acting in the hyperbolic space $\mathbb{H}^n$ associated to $N^1(X)_\mathbb{R}$.

Note that surface X is an example of a K3 surface.

Example 6.2. There is another example, where the group $\mathsf{UC}(3)$ is realized as a discrete group of motions in $\mathbb{H}^3$ associated with automorphisms of a K3 surface. We consider nonsingular surface X given in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by an equation $F(u_0, u_1; v_0, v_1; w_0, w_1)$ which is multi-homogeneous of degree 2 in coordinates on each copy of $\mathbb{P}^1$. It is known that the canonical class of X is equal to 0, and the first Betti number of X is equal to 0, so X must be a K3 surface. Consider the projection p_{ij} to the product of the ith and jth factors. Assume for simplicity $(i, j) = (1, 2)$. If we write

$$F = w_0^2 A_1 + 2w_0 w_1 A_2 + w_1^2 A_3, \tag{6.5}$$

where A_i are bihomogeneous forms of degree $(2, 2)$ in (u_0, u_1, v_0, v_1). The fiber over a point (x, y) consists of two points taken with multiplicity. So, the cover $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is of degree 2 ramified over the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying the equations $A_1 A_3 - A_2^2 = 0$. Let g_{ij} be the deck transformation of this cover. Let us see how it acts on $N^1(X)$. There are obvious classes h_i in $N^1(X)$ represented by the fibers of the projections $p_i : X \to \mathbb{P}^1, i = 1, 2, 3$. The fiber is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ given by a bihomogeneous equation of degrees $(2,2)$. We have

$$((h_i, h_j)) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Let us consider the span M of h_1, h_2, h_3 in $N^1(X)$. It follows from formula (6.5) that h_i, h_j is invariant with respect to g_{ij}^*, and $h_k + g_{ij}^*(h_k) = 2h_i + 2h_j$. Thus, we see that each g_{ij}^*, say g_{12}^*, is defined in the basis (h_1, h_2, h_3) by the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is a reflection with respect to the vector $\alpha = h_1 + h_2 - h_3$ of norm -4. The group generated by $g_{12}^*, g_{13}^*, g_{23}^*$ is isomorphic to the universal Coxeter group $\mathsf{UC}(3)$.

Example 6.3. Similarly to the previous example, we can realize the group $\mathsf{UC}(4)$ as acting on a K3 surface given as a smooth complete intersection of two divisors of types $(1,1,1,1)$. If X is general enough, then $N^1(X)$ is equal to the restriction of $\mathrm{Pic}((\mathbb{P}^1)^4)$ to X. The projection p_{ij} to the product of any two factors is a double cover (its fibers are equal to the intersection of two divisors of type $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$). We realize $\mathsf{UC}(4)$ as the group generated by the deck transformations of these covers.

The fundamental polyhedron for $\mathsf{UC}(3)$ is the ideal triangle, its orbit under the reflection group gives a tesselation of the hyperbolic plane.

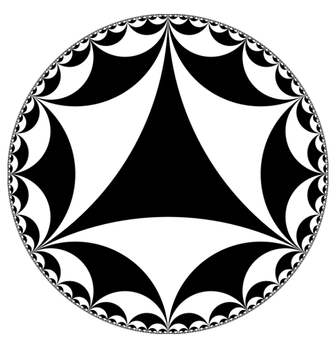


Figure 6.1: Ideal triangle

The importance of the reflection groups in algebraic geometry is explained by the following fact that follows from the *Global Torelli Theorem* for K3 surfaces due to I.R. Shafarevich and I.I. Piyatetski-Shapiro.

Theorem 6.4. *Assume $\Bbbk = \mathbb{C}$. Let X be a K3 surface or an abelian surface. Let $\sigma : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ be an isometry of the quadratic lattice $H^2(X,\mathbb{Z})$. Suppose that, $\sigma^*(H^{2,0}(X)) = H^{2,0}(X)$, and, for any effective divisor class $[C]$ on X, $\sigma([C])$ is an effective divisor class.*[1] *Then there exists an automorphism g of X such that $g^* = \sigma$.*

The Riemann-Roch Theorem applied to K3 surfaces implies that for any divisor class D with $D^2 \geq -2$, we have either D or $-D$ is effective. We know that the set $P_X = \{x \in N^1(X)_{\mathbb{R}} : x^2 > 0\}$ consists of two connected components. Choose one component P_X^+ that contains the divisor class h of a hyperplane section for some projective embedding of X. Then all effective divisor classes are contained in this component. We know that the group of isometries $N^1(X)_{\mathbb{R}}$ preserving P_X^+ coincides with the group of motions of the hyperbolic space $\mathbb{P}(P_X^+) \cong \mathbb{H}^{\rho(X)-1}$ associated with $N^1(X)$. Let W_X denote the group of isometries of P_X^+ generated by reflections s_r, where r is the class of divisor class R with $R^2 = -2$. One can show that this group coincides with the group generated by reflections s_r, where r is the class of a smooth rational curve R on X. Note that in the case of an abelian surface no such curve exists on X (since a holomorphic map of $\mathbb{P}^1$ to a torus $\mathbb{C}^2/\lambda$ lifts to a holomorphic map from $\mathbb{P}^1$ to $\mathbb{C}^2$, and hence must be constant). We call such classes *nodal roots*. Let $\mathcal{C}_X$ be the fundamental domain of W_X in P_X^+. It is the cone over the fundamental domain of W_X in $\mathbb{P}(P_X^+)$. Writing any effective divisor as a non-negative linear combination of irreducible curves, we see that $[D] \in \mathcal{C}_X$ if and only if D is nef, i.e. intersects any curve non-negatively.

Let $\mathrm{O}(N^1(X))^+$ be the subgroup of $\mathrm{O}(N^1(X)$ that preserves $\mathcal{C}_X$. It is easy to see that any isometry of $\mathrm{O}(N^1(X))$ that acts identically on the discriminant group of $N^1(X)$ extends to an isometry of $H^2(X,\mathbb{Z})$ that acts on the orthogonal complement of $N^1(X)$ identically. Thus $\mathrm{O}(N^1(X))^+$ contains a subgroup of finite index whose elements are extended to such isometries of $H^2(X,\mathbb{Z})$. Obviously they preserve $H^{2,0}(X)$ (since it is contained in the orthogonal complement of $H^{1,1}(X)$). Since any element α of $\mathrm{O}(N^1(X))^+$ preserves P_X^+, for any effective divisor class D with $D^2 \geq 0$, we have $\alpha(D)$ is effective. For any root r, $\alpha(r)$ or $-\alpha(r)$ is effective. If the second possibility realizes, then $0 > \alpha(r) \cdot h = r \cdot \alpha^{-1}(h)$. Since $h \in \mathcal{C}_X$, the divisor class $\alpha^{-1}(h) \in \mathcal{C}_X$, hence it intersects any effective divisor non-negatively. This contradiction shows that $\alpha(r)$ is effective. Aplying the Global Torelli Theorem, we obtain the following corollary of the Global Torelli Theorem.

Corollary 6.5. *The subgroup of $\mathrm{O}(N^(X))$ generated by W_X and $\mathrm{Aut}(X)^*$ is of finite index.*

Note that $W_X \cap \mathrm{Aut}(X)^* = \{1\}$ and W_X is a normal subgroup of the group generated by W_X and $\mathrm{Aut}(X)^*$. Thus the latter group is isomorphic to the semi-direct product $W_X \rtimes \mathrm{Aut}(X)^*$.

Corollary 6.6. *The group of automorphisms of a K3 surface is finite if and only if the lattice $N^1(X)$ is 2-reflective. The group of connected components of the automorphism group of an abelian surface is finite if and only if $\mathrm{O}(N^1(X))$ is finite.*

[1] To satisfy this condition it is enough to assume that σ sends any *ample divisor class* to an ample divisor class. An ample divisor class is the class of a divisor such that its positive multiple defined a projective embedding of the surface.

Example 6.7. All 2-reflective quadratic lattices M of signature $(1, n)$ with $n \neq 3$ have been classified by V. Nikulin and by E. Vinberg (in the case $n = 4$) (see [41], [42], [55]). They are all realized as the Picard lattices $N^1(X)$ for some K3 surface. The largest possible rank of such a lattice is equal to 19. The Coxeter diagram of the 2-reflection group $\mathrm{Ref}_2(M)$ in this case is the following one.

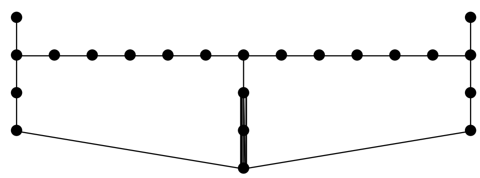


Figure 6.2: 2-reflective lattice of rank 19

We have an isomorphism of lattices

$$M \cong \mathsf{U} \perp \mathsf{E}_8 \perp \mathsf{E}_8 \perp \mathsf{A}_1.$$

The K3 surfaces whose Picard group contains the lattice M are described in [17].

There is an analog of Corollaries 6.5 and 6.6 for Enriques surfaces. In this case not each vector in $N^1(X)$ with norm -2 is effective. Note that $N^1(X) \cong \mathsf{E}_{10}$ and $\mathrm{Ref}_2(\mathrm{O}(\mathsf{E}_{10}))$ is a subgroup of index 2 of $\mathrm{O}(\mathsf{E}_{10})$. Let W_X denote the subgroup of $\mathrm{O}(N^1(X))$ generated by reflections in the classes r of smooth rational curves on X. A general Enriques surface does not contain smooth rational curves, so this group is trivial.

Corollary 6.8. *The subgroup of* $\mathrm{O}(N^(X))$ *generated by* W_X *and* $\mathrm{Aut}(X)^*$ *is of finite index.*

Note that $W_X \cap \mathrm{Aut}(X)^* = \{1\}$ and W_X is a normal subgroup of the group generated by W_X and $\mathrm{Aut}(X)^*$. Thus the latter group is isomorphic to the semi-direct product $W_X \rtimes \mathrm{Aut}(X)^*$.

Corollary 6.9. *The group of automorphisms of an Enriques surface is finite if and only if the lattice* $N^1(X)$ *admits a crystallographic root basis formed by the classes of rational smooth curves.*

One can show that all the previous assertions are true in the case of an arbitrary algebraically closed field $\Bbbk$. Over $\mathbb{C}$, there is a classification due to S. Kondo and V. Nikulin of all possible Enriques surfaces with finite automorphism groups. They are divided into 7 types according to a possible group of automorphisms and a possible crystallographic root basis formed by smooth rational curves. In Figure 5.3 we gave examples of crystallographic root bases in the lattice E_{10} of cardiianlity ≤ 12. The first three examples are realized only if $\Bbbk$ is of characteristic 2. The last example is an example of Type 1 on Kondo-Nilulin's classification. It was first constructed in [16]. The group of automorphisms of the corresponding Enriques surfaces is isomorphic to the dihedral group D_8 of order 8.

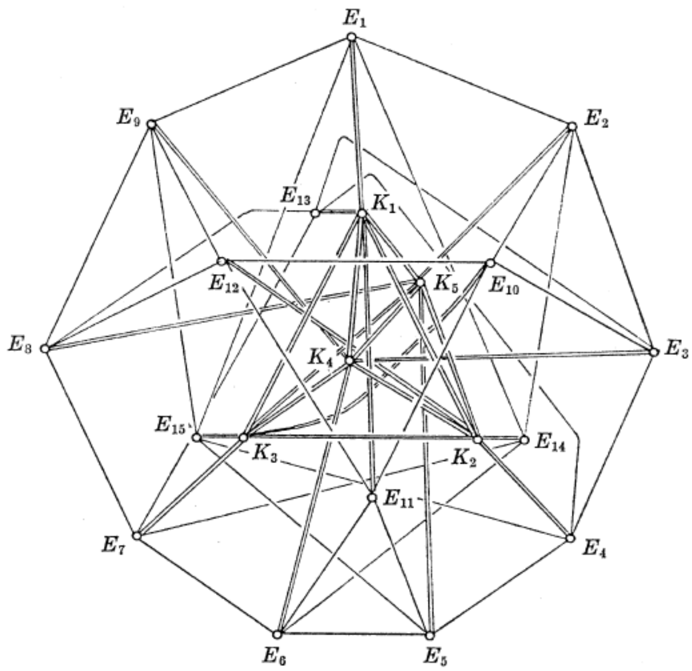


Figure 6.3: Crystallographic root basis of cardinality 20

The following example is of type VII. It was first discovered by G. Fano. The group of automorphisms of the corresponding Enriques surfaces is isomorphic to the permutation group $\mathfrak{S}_5$.

Mysteriously, all crystallographic root bases in types I-VII are of cardinality 12 ?r 20.

Example 6.10. Let A be an abelian surface. The group $\mathrm{Aut}(A)_0$ is isomorphic to A, where the latter acts on itself by translations. The group of connected components $\mathrm{Aut}_c(A)$ is isomorphic to the group automorphisms of A, the invertible elements in the ring $\mathrm{End}(A)$ of endomorphisms of A. An abelian surface is called *simple* if does not map surjectively to the product of elliptic curves. Let $\mathrm{End}(A)_{\mathbb{Q}} = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{R}$. It is an algebra over $\mathbb{Q}$. The classification of possible structures of the ring of automorphisms of *abelian varieties* (complex tori of arbitrary dimension embeddable in a projective space) shows that there are the following possibilities for the structure of $\mathrm{End}(A)_{\mathbb{Q}}$ when A is a simple abelian surface.

(i) $\mathrm{End}(A)_{\mathbb{Q}} \cong \mathbb{Q}$ and $\rho(A) = 1$;

(ii) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally real quadratic extension K of $\mathbb{Q}$ and $\rho(A) = 2$;

(iii) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally indefinite quaternion algebra over $K = \mathbb{Q}$ and $\rho = 3$;

(iv) $\mathrm{End}(A)_{\mathbb{Q}}$ is a totally imaginary quadratic extension K of a real quadratic field K_0 and $\rho = 2$.

As we see that $\mathrm{End}(A)_{\mathbb{Q}}$ is a central simple algebra, so that $\mathrm{End}(A)$ is an order in it. It follows from Dirichlet's Theorem on units in an order of algebraic integers that $\mathrm{Aut}_c(A)$ could be an infinite group only in the case (iii) and (iv). In the former case the group of units $\mathrm{End}(A)^* = \mathrm{Aut}_c(A)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. It is a cocompact *Fuchsian group* of the first kind . It is isomorphic to a discrete cocompact subgroup of $\mathbb{H}^2$ with the limit set equal to the whole absolute. In the last case (iv), one says that an abelian surface has *complex multiplications*. The group of units is an abelian group of rank 1. The quadratic from of the lattice $N^1(A)$ is an indefinite lattice of rank 2 not representing zero. Its orthogonal group is infinite and contains $\mathrm{Aut}(A)^*$ as its subgroup of finite index.

Assume now that A is not simple. In this case $\mathrm{End}(A)_{\mathbb{Q}} \cong \mathrm{End}_Q(E_1 \times E_2)$, where E_1, E_2 are elliptic curves. If $E_1 \not\cong E_2$, then $\rho(A) = 2$ and $\mathrm{O}(N^1(A))$ is a finite group. If $E_1 \cong E_2$, then $\rho(A) \geq 3$. It is equal to 4 if and only if E has *complex multiplication*, i.e. $\mathrm{End}(E)$ is an order in an imaginary quadratic field $K/\mathbb{Q}$. It is easy to see that $\mathrm{Aut}_c(A)$ is a subgroup of $\mathrm{Aut}_c(E \times E)$. Suppose that $\mathrm{End}(E) = \mathbb{Z}$ and let $g \in \mathrm{Aut}_c(E \times E)$. Then the image of the curve $E \times \{0\}$ in $E \times E$ projects surjectively to each factor and defines a pair (m, n) of endomorphisms $E \to E$. It follows that g can be represented by an invertible integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Conversely, each matrix defines an automorphism of $E \times E$. Thus we obtain

$$\mathrm{Aut}_c(E \times E) \cong \mathrm{GL}_2(\mathbb{Z}).$$

Note that the automorphism $(-1, -1)$ acts identically on the lattice $N^1(X)$ of rank 3, so that

$$\Gamma := \mathrm{Aut}(E \times E)^* \cap \mathrm{Iso}(\mathbb{H}^2)^+ \cong \mathrm{PSL}_2(\mathbb{Z}).$$

Now we assume that E has a complex multiplication with an imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-m})$, where m is a positive square free integer. Let $\mathfrak{o} = \mathrm{End}(E)$, it is an order in K. The elliptic curve is isomorphic to the complex torus $\mathbb{C}/\mathfrak{o}$, where $\mathfrak{o} = \mathbb{Z}f_1 + \mathbb{Z}f_2$, where (f_1, f_2) is a basis in $\mathfrak{o}$. Arguing as before, we obtain that

$$\mathrm{Aut}(E \times E) \cong \mathrm{GL}(2, \mathfrak{o}),$$

$$\Gamma := \mathrm{Aut}(E \times E) \cap \mathrm{Iso}(\mathbb{H}^3)^+ \cong \mathrm{PSL}_2(\mathfrak{o}).$$

Suppose $\mathfrak{o} = \mathfrak{o}_K$ is the total order, i.e. coincides with the ring of integers of K. It is known that one can choose a basis in $\mathfrak{o}_K$ of the form $(1, \omega)$, where $\omega = \sqrt{-m}$ if $-m \not\equiv 1 \mod 4$ and $\omega = \frac{1}{2}(1 + \sqrt{-m})$ otherwise.

One can show that we have an isomorphism of lattices

$$N^1(X) \cong U \oplus \langle -2 \rangle \oplus \langle -m \rangle, \text{ if } -m \not\equiv 1 \mod 4.$$

$$N^1(X) \cong U \oplus \left(\begin{smallmatrix} -2 & 1 \\ 1 & -m \end{smallmatrix}\right), \text{ if } -m \equiv 1 \mod 4.$$

The group Γ is a *Bianchi group* $\mathrm{Bi}(m)$. As we noticed before, it is always a lattice in $\mathrm{Iso}(\mathbb{H}^3)^+ \cong \mathrm{PSL}_2(\mathbb{C})$ but not cocompact. Let $\mathrm{Bi}(m)_r$ be the maximal reflection subgroup of $\mathrm{Bi}(m)$. It is a lattice in $\mathrm{Iso}(\mathbb{H}^3)$ if and only if $m \leq 19, m \neq 14, 17$ [5]

Lecture 7

Boyd-Maxwell Coxeter groups and sphere packings

We know that one can identify the absolute $\partial\mathbb{H}^n$ with the $n-1$-dimensional sphere S^{n-1}, and, via the stereographic projection, with the extended Euclidean space $\hat{E}^{n-1}$. Let $H_{\mathfrak{e}}$ be a hyperplane in $\mathbb{H}^n$ with the normal vector $\mathfrak{e}$ of norm -1. Its closure in $\overline{\mathbb{H}}^n$ intersects the boundary, let us see that it cuts out a sphere in $\hat{E}^{n-1}$ (maybe of infinite radius).

For convenience, and future applications, let us change the standard coordinates and take the new coordinates in $\mathbb{R}^{1,n}$ such that the absolute has the equation

$$q = 2t_0t_n - t_1^2 - \cdots - t_{n-1}^2 = 0. \tag{7.1}$$

Let

$$\phi : \mathbb{P}^{n-1}(\mathbb{R}) \dashrightarrow \mathbb{P}^n(\mathbb{R}) \tag{7.2}$$

be the map defined by the formula

$$[x_0, \ldots, x_{n-1}] \mapsto [2x_0^2, -2x_0x_1, \ldots, -2x_0x_{n-1}, \sum_{i=1}^{n-1} x_i^2].$$

It is immediately checked that the map is everywhere defined in $\mathbb{P}^{n-1}(\mathbb{R})$ (but not in $\mathbb{P}^{n-1}(\mathbb{C})$). The image of the hyperplane $x_0 = 0$ is the point $p = [0, 0, \ldots, 0, 1] \in Q$ which we may call the north pole. The complement $\mathbb{P}^{n-1}(\mathbb{R}) \setminus \{x_0 = 0\}$ can be identified with the affine space $E^{n-1} = \mathbb{R}^{n-1}$ via dehomogenization of the coordinates $y_i = x_i/x_0, i = 1, \ldots, n-1$. Its image is equal to $Q \setminus \{p\}$, and the inverse map $Q \setminus \{p\} \to E^{n-1}$ is the projection map from p to the plane $x_0 = 0$. So, this gives an explicit identification of the absolute with $\hat{E}^{n-1} = E^{n-1} \cup \{p\}$.

Let $\mathfrak{e} = (a_0, \ldots, a_n) \in \mathbb{R}^{1,n}$ with $(\mathfrak{e}, \mathfrak{e}) = -1$. The pre-image of a hyperplane

$$H_{\mathfrak{e}} = \{x \in \mathbb{R}^{1,n} : (x, \mathfrak{e}) = 0, (x, x) \geq 0\}/\mathbb{R}^* \subset \mathbb{P}^n(\mathbb{R})$$

under the map ϕ is a quadric in $\mathbb{P}^{n-1}(\mathbb{R})$ defined by the equation

$$a_0\sum_{i=1}^{n-1} x_i^2 + 2a_nx_0^2 - 2x_0\sum_{i=1}^{n-1} a_ix_i = 0.$$

If $a_0 \neq 0$, this quadric does not intersect the hyperplane $x_0 = 0$. We can rewrite this equation in the form

$$a_0^2\sum_{i=1}^{n-1}(\frac{x_i}{x_0} - \frac{a_i}{a_0})^2 = -2a_0a_n + \sum_{i=1}^{n-1} a_i^2 = -(\mathfrak{e},\mathfrak{e}) = 1. \tag{7.3}$$

So, we can identify its real points with a $n-1$-dimensional sphere $S(\mathfrak{e})$ in the Euclidean space $E^{n-1} = \mathbb{P}^{n-1}(\mathbb{R}) \setminus \{x_0 = 0\}$ of radius square $r^2 = 1/a_0^2$ and the center $c = [\frac{a_1}{a_0}, \ldots, \frac{a_{n-1}}{a_0}]$. It is natural to call $|a_0|$ the *curvature* of the sphere.

If $a_0 = 0$, the quadric is equal to the union of the hyperplane at infinity and the hyperplane

$$S(\mathfrak{e}) := \sum_{i=1}^{n-1} a_ix_i + a_{n+1}x_0 = 0.$$

We view this hyperplane as a sphere with curvature $k = 0$ (or radius $r := \infty$).

One introduces the oriented curvature equal to a_0. We agree that the positive curvature corresponds to the interior of the sphere $S(\mathfrak{e})$, i.e. an open ball $B(\mathfrak{e})$ of radius r. The negative curvature corresponds to the open exterior of the sphere, we also call it the *ball* corresponding to the oriented sphere S and continue to denote it $B(\mathfrak{e})$. It can be considered as a ball in the extended Euclidean space $\hat{E}^{n-1}$.

So, we see that the mirror hyperplanes of reflections s_e intersect the absolute along a sphere, if we identify the absolute with the extended Euclidean space $\hat{E}^{n-1}$.

A *sphere packing* in the extended Euclideal space $\hat{E}^k$ is an infinite set $\mathfrak{P} = (S_i)_{i\in I}$ of oriented k-spheres such that any two of them are either disjoint or touch each other (i.e. intersect at one point). We say that a sphere packing is *strict* if, additionally, no two open balls B_i intersect. An example of a non-strict sphere packing is an infinite set of nested spheres. A sphere-packing is called *maximal* if any sphere overlaps with one of the spheres from $\mathfrak{P}$.

We assume also that the set $\mathfrak{P}$ is *locally finite* in the sense that, for any $t > 0$, there exists only finitely many spheres of curvature at most t in any fixed bounded region of the space.

The condition that two spheres S_i and S_j are disjoint or touch each other is easily expressed in terms of linear algebra. We have the following.

Lemma 7.1. *Let $S(\mathfrak{e})$ and $S(\mathfrak{e}')$ be two n-dimensional spheres corresponding to hyperplanes $H_{\mathfrak{e}}$ and $H_{\mathfrak{e}'}$. Then their interiors do not intersect if and only if*

$$(\mathfrak{e},\mathfrak{e}') \geq 1.$$

The equality takes place if and only if the spheres are tangent to each other, and hence intersect at one real point.

Proof. Suppose $(\mathfrak{e}, \mathfrak{e}') \geq 1$, then the plane spanned by $\mathfrak{e}, \mathfrak{e}'$ is not negative definite, hence its orthogonal complement does not contain vectors with positive norm. This implies that the hyperplanes $H_{\mathfrak{e}}$ and $H_{\mathfrak{e}'}$ either diverge or, if $(\mathfrak{e}, \mathfrak{e}') = 1$, intersect at one point in the absolute corresponding to the unique isotropic line in the orthogonal complement of the plane. The converse is proven along the same lines. □

Let P be a *simplicial Coxeter polytope* in $\mathbb{H}^n$ defined by a *Coxeter polytope!simplicial* in $\mathbb{R}^{1,n}$. This means that it is bounded by $n+1$ hyperplanes $H_{\mathfrak{e}_i}$, where $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1})$ is a basis in $\mathbb{R}^{1,n}$.

Let ω_j be a vector in $\mathbb{R}^{1,n}$ uniquely determined by the condition

$$(\omega_j, \mathfrak{e}_i) = \delta_{ij}. \tag{7.4}$$

We have

$$\omega_j = \sum_{i=1}^{n+1} g^{ij} \mathfrak{e}_i, \tag{7.5}$$

where

$$G(P)^{-1} = (g^{ij}) = ((\omega_i, \omega_j)).$$

The vectors $\omega_1, \ldots, \omega_{n+1}$ are called *fundamental weights* associated to *simple roots* $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1})$. We call ω_i *real* if $(\omega_i, \omega_i) = g^{ii} > 0$. In this case we can normalize it to set

$$\bar{\omega}_i := \omega_i / \sqrt{g^{ii}}.$$

Let $J = \{j_1 < \cdots < j_r\}$ be the subset of $\{1, \ldots, n+1\}$ such that $\bar{\omega}_j$ is real if and only if $j \in J$. Consider the union

$$\mathfrak{P} = \bigcup_{k=1}^{r} O_{\Gamma_P}(S(\bar{\omega}_{j_k})). \tag{7.6}$$

A *Boyd-Maxwell sphere packing* is a sphere packing of the form $\mathfrak{P}(P)$, where P is a simplicial Coxeter polytope. By definition, it is clustered with *clusters*

$$(\gamma(S(\bar{\omega}_{j_1})), \ldots, \gamma(S(\bar{\omega}_{j_r}))) = (S(\gamma(\bar{\omega}_{j_1})), \ldots, S(\gamma(\bar{\omega}_{j_r})).$$

We call the cluster $(S(\omega_{j_1}), \ldots, S(\omega_{j_r}))$ the *initial cluster*. It is called *non-degenerate* if all ω_i are real. This is equivalent to that all principal maximal minors of the matrix $G(P)$ are negative. It is clear that the spheres in the packing $\mathfrak{P}(P)$ correspond to elements of the Γ_P-orbit of real fundamental roots. We call them *real weights*.

Let Δ_{ij} be the minors of $G(P)$ obtained by deleting the ith row and j-th column. Let $\Delta = |G(P)|$. Its sign is the same as the sign of the of the determinant of the matrix J_{n+1} defining the quadratic form (7.1). So, it is equal to $(-1)^n$. Since

$$(\omega_i, \omega_j) = (-1)^{i+j} \Delta_{ij} / \Delta, \tag{7.7}$$

we obtain that ω_i is real if and only if $(-1)^{n-1}\Delta_{ii} < 0$.

Following G. Maxwell [38], we say that the Coxeter diagram is of *level* l if, after deleting any l of its vertices, we obtain a Coxeter diagram of Euclidean or of parabolic type describing a Coxeter polyhedron in an Euclidean or a spherical geometry. They can be characterized by the property that all $m_{ij} \neq \infty$ (except in the case $n = 1$) and the symmetric matrix (m_{ij}) is non-negative definite, definite in the Euclidean case, and having a one-dimensional radical in the parabolic case. Coxeter diagrams of level $l = 1$ correspond to Lanner and quasi-Lanner Coxeter groups. Coxeter diagrams of level 2 have been classified by G. Maxwell (with three graphs omitted, see [12]). They have $N \leq 11$ vertices. For $N \geq 5$ they are obtained from quasi-Lanner diagrams by adding one vertex. We call Coxeter polytopes with Coxeter diagram of level 2 a *Boyd-Maxweel* polytope.

For example, a Coxeter polyhedron in $\mathbb{H}^{10}$ with Coxeter diagram of type $T_{2,3,8}$ is obtained from the diagram $T_{2,3,7}$ by adding one vertex v is of level 2

It defines a sphere packing with only one real ω_i corresponding to the vertex v.

The following theorem is proven, under a certain assumption, in loc.cit., Theorem 3.3. The assumption had been later removed in [39], Theorem 6.1.

Theorem 7.2. *Let P be a Coxeter polytope in $\mathbb{H}^n$. Then $\mathfrak{P}(P)$ is a maximal sphere Γ_P-packing in $\hat{E}^{n-1}$ if and only if P is a Boyd-Maxwell polytope. It is a non-strict sphere packing if and only if there exists a pair of distinct vertices corresponding to real fundamental weights such that, after deleting them, one obtains a Coxeter diagram of finite type.*

If P is a Coxeter polytope of quasi-Lanner type, the group Γ_P has a fundamental polyhedron of finite volume. It follows that the index 2 subgroup of Γ_P is a Kleinian group of the first kind ([45], Theorem 12.2.13). Thus any point on the absolute is its limit point. The stabilizer $\Gamma_{P,i}$ of each $S(\bar{\omega}_i)$ is the reflection group of the Coxeter group defined by the Coxeter diagram of level 1, hence the limit set of $\Gamma_{P,i}$ is equal to $S(\bar{\omega}_i)$. Thus the assumption (A1) is satisfied.

Let $\mathfrak{P}(P)$ be a non-degenerate Boyd-Maxwell sphere packing and let $\mathbf{k} = (k_1, \ldots, k_{n+1})$ be the vector of the curvatures of the spheres

$$S(\gamma(\omega_1)),\ \ldots,\ S(\gamma(\omega_{n+1}))$$

in its cluster. Let N be the diagonal matrix diag$(\sqrt{-g^{11}}, \ldots, \sqrt{-g^{n+1}})$. Denote by $\widetilde{G(P)}$ the matrix $NG(P)N$.

Theorem 7.3. *Let $G(P)$ be the Gram matrix of P. Then*

$$^t\mathbf{k} \cdot \widetilde{G(P)} \cdot \mathbf{k} = 0. \tag{7.8}$$

Proof. Obviously $\widetilde{G(P)} = \widetilde{G(\gamma(P)})$. So, we may assume that the cluster $S(\gamma(\omega_1), \ldots, S(\gamma(\omega_{n+1}))$ is equal to the initial cluster $S(\omega_1), \ldots, S(\omega_{n+1})$.

Let J_{n+1} be the matrix of the symmetric bilinear form defined by the fundamental quadratic form q from (7.1). It satisfies $J_{n+1} = J_{n+1}^{-1}$. Let X be the matrix whose jth column is the vector of coordinates of the vector $\bar{\omega}_j$. Recall from (7.3) that the first coordinate of each vector $\bar{\omega}_j$ is equal to the curvature of the sphere $S(\bar{\omega}_j)$. By definition of the Gram matrix, we have

$$^{t}X \cdot J_{n+1} \cdot X = ((\bar{\omega}_i, \bar{\omega}_j)) = N^{-1}G(P)^{-1}N^{-1},$$

hence

$$^{t}X^{-1} \cdot N^{-1}G(P)^{-1}N^{-1} \cdot X^{-1} = J_{n+1}.$$

Taking the inverse, we obtain

$$X \cdot \widetilde{G(P)} \cdot {}^{t}X = J_{n+1}^{-1} = J_{n+1}.$$

The first entry a_{11} of the matrix in the right-hand side is equal to zero. Hence

$$^{t}\mathbf{k} \cdot \widetilde{G(P)} \cdot \mathbf{k} = 0,$$

□

The group Γ_P generated by the reflections $s_i := s_{\mathfrak{e}_i}$ acts on the dual basis $(\omega_1, \ldots, \omega_{n+1})$ by the formula

$$s_i(\omega_j) = \omega_j - 2(e_i, \omega_j)\mathfrak{e}_i = \omega_j - 2\delta_{ij}\mathfrak{e}_i = \omega_j - 2\delta_{ij}\sum_{k=1}^{n+1} g_{ki}\omega_k, \tag{7.9}$$

where δ_{ij} is the Kronecker symbol. This gives explicitly the action of Γ_P on the clusters of the sphere packing and also on the set of their curvature vectors.

Note that, in general, the polytope defined by the hyperplanes $H_{\bar{\omega}_i}$ is not a Coxeter polytope. If it is, and also a Boyd-Maxwell polytope, then one may define the dual sphere packing.

Note that different Boyd-Maxwell polytopes may define the same sphere packings. Thus 186 possible Coxeter diagrams of level 2 with $N \geq 5$ vertices define only 95 different sphere packings.

Theorem 7.4. *Let P be a Boyd-Maxwell polytope. Then the limit set $\Lambda(\Gamma_P)$ is equal to the closure of the union of spheres from the packing.*

Proof. It follows from Theorem of [39] that the closure of the union of spheres in the packing $\mathfrak{P}(P)$ is equal to the closure of the union of orbits of spheres corresponding to real fundamental weights. Let S_i be a sphere from the packing $\mathfrak{P}(P)$ from the orbit of a real fundamental weight and let Γ_i be its stabilizer subgroup in Γ_P. We view S_i as the absolute in the codimension 1 hyperbolic subspace $\mathbb{H}^{n-1}$ of $\mathbb{H}^n$. Since deleting the vertex corresponding to a real fundamental weight is a subdiagram of quasi-Lanner type, the group Γ_i is of finite covolume in $\mathbb{H}^{n-2}$, and its limit set if equal to the absolute. Thus S_i is equal to the closure of fixed points of hyperbolic and parabolic elements in Γ_i. Suppose that $x \in \partial\mathbb{H}^n$ is a fixed point of a hyperbolic or a parabolic element γ from Γ_P that does not belong to any sphere from the packing. Since the sphere packing is maximal, it belongs to the interior of some sphere from the packing and leaves it invariant. But then γ must be of finite order contradicting the assumption. This shows that the closure of the union of spheres is equal to the closure of fixed points of hyperbolic and parabolic elements in Γ_P, and hence coincides with the limit set. □

Example 7.5. The most notorious and widely discussed beautiful example of a Boyd-Maxwell sphere packing is the *Apollonian cicle packing* (see, for example, [27], [28], [29], [26], [47]).

Consider again the universal Coxeter group $\mathsf{UC}(n+1)$ defined by the Coxeter polytope with the Gram matrix of size $n+1$

$$G(P) = \mathrm{circ}(-1, 1, \ldots, 1). \tag{7.10}$$

Its Coxeter diagram is the complete graph with $n+1$ vertices and thick edges. It is of level 2 only if $n = 3$. Thus it defines a maximal sphere packing in the plane $\hat{E}^2$. This is the Apollonian circle packing.

It is easy to compute the inverse of $G(P)$ to obtain

$$G(P)^{-1} = \frac{n-2}{2n-2}\mathrm{circ}(-1, \frac{1}{n-2}, \ldots, \frac{1}{n-2}). \tag{7.11}$$

It is the Gram matrix of a Coxeter polytope P if and only if $n = 3, 4$. So, we may consider a polytope $P^\perp$ with $G(P^\perp) = \frac{2n-2}{n-2}G(P)^{-1}$ and define a sphere-packing in $\hat{E}^3$. If $n > 4$, we can still use this matrix but the group $\Gamma_{P^\perp}$ is not a discrete anymore, so the sphere packing is not locally finite anymore.

The group $\mathsf{UC}(n+1)$ realized as the reflection group with the polytope with the Gram matrix (7.10) is called in literature the *Apollonian group*. We call P the *Apollonian polyhedron*. It is denoted by Ap_n. Note that when $n = 3$, the normalized fundamental weights of P define the same Gram matrix as the simple roots of P but they different from the roots. The corresponding sphere packings are orthogonal to each other as shown in the following picture. We consider the polytope $P^\perp$. Since g^{ii} are all equal, the matrix N in equation (7.8) is scalar, hence $\widetilde{G(P^\perp)} = G(P^\perp)$ and we curvature vector satisfies the equation

$$n(k_1^2 + \cdots + k_{n+1}^2) - (k_1 + \cdots + k_{n+1})^2 = 0. \tag{7.12}$$

It is known as *Descartes's equation* or *Soddy's equation*[1]

Let $H_{\bar{\omega}_1}, \ldots, H_{\bar{\omega}_{n+1}}$ be the bounding hyperplanes for $P^\perp$ and let $S(\mathfrak{e}_1), \ldots, S(\mathfrak{e}_{n+1})$ be the cluster of the dual Apollonian sphere packing. The formula (7.9) specializes to give $s_{e_i}(\bar{\omega}_j) = \bar{\omega}_j$,

[1] From a "poem proof" of the theorem in the case $n = 3$ in "Kiss Precise" by Frederick Soddy published in Nature, 1930:

Four circles to the kissing come. / The smaller are the bender. / The bend is just the inverse of / The distance from the center. / Though their intrigue left Euclid dumb / Theres now no need for rule of thumb. / Since zero bends a dead straight line / And concave bends have minus sign, / The sum of the squares of all four bends / Is half the square of their sum.

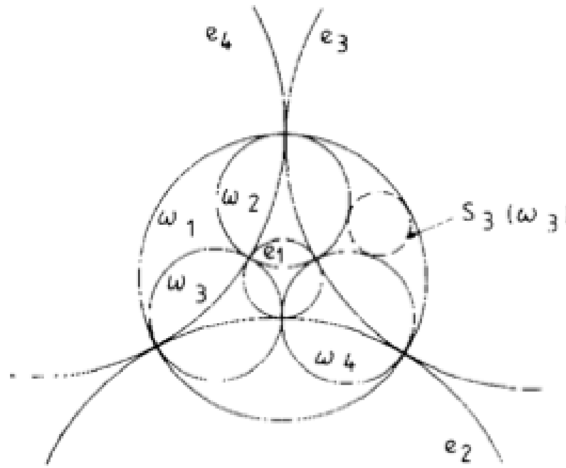


Figure 7.1: Dual Apollonian circle clusters (from [38])

and

$$s_{\mathfrak{e}_i}(\bar{\omega}_i) = -\bar{\omega}_i + \frac{2}{n-1}\sum_{j\neq i}\bar{w}_j.$$

For example, $s_{\mathfrak{e}_1}$ is represented in the basis $(\bar{\omega}_1, \ldots, \bar{\omega}_{n+1})$ by the matrix

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 \\ \frac{2}{n-1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{2}{n-1} & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

So, the Apollonian group is generated by $n+1$ matrices $A_1, \ldots, A_{n+1}$ of this sort.

Assume $n > 2$. Consider a solution $(k_1, \ldots, k_{n+1})$ of the Descartes' equation which we rewrite in the form

$$\sum_{i=1}^{n+1} k_i^2 - \frac{2}{n-2}\sum_{1\leq i\leq j\leq n+1} k_i k_j = 0. \tag{7.13}$$

Thus, k_{n+1} satisfies the quadratic equation

$$t^2 - \frac{2t}{n-2}\sum_{i=1}^{n} k_i - \frac{2}{n-2}\sum_{1\leq i<k\leq n}^{n+1} k_i = 0.$$

It expresses the well-know fact (the *Apollonian Theorem*) that, given $n+1$ spheres touching each other, there are two more spheres that touch them. Thus, starting from an Apollonian cluster

$(S_1, \ldots, S_{n+1})$, we get a new Apollonian cluster $(S_1, \ldots, S_{n+1}, S'_{n+1})$ such that the curvatures of S_{n+1} and S'_{n+1} are solution of the above quadratic equation. The curvatures of S'_{n+1} is equal to

$$k'_{n+1} = -k_{n+1} + \frac{2}{n-2}\sum_{i=1}^{n} k_i.$$

So the new cluster $(S_1, \ldots, S_{n+1}, S'_{n+1})$ is the cluster obtained from the original one by applying the transformation $s_{e_{n+1}}$.

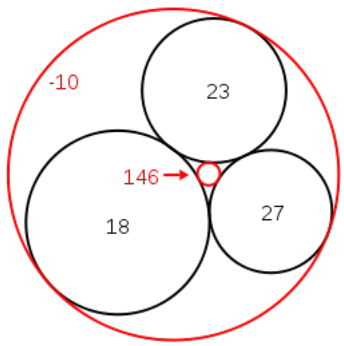


Figure 7.2: Apollonina circle cluster with curvature vector $(-10, 18, 23, 27)$

If we start with a cluster as in the picture, then all circles will be inclosed in a unique circle of largest radius, so our circle packing will look as in following Figure 7.3.

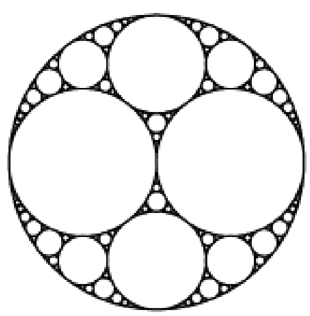


Figure 7.3: Apollonina circle packing

We have already considered the Apollonian group $\mathrm{Ap}_2 = \mathsf{UC}(3)$ in Example 6.2. One can consider the analog of the Apollonian packing in this case as the set of not overlapping intervals on the unit circle with ends equal to the sides of the images of an ideal triangle under the reflection group.

The Descartes equation becomes

$$k_1k_2 + k_1k_3 + k_2k_3 = 0. \tag{7.14}$$

Let $K \subset \mathbb{R}$ be a totally real field of algebraic numbers of degree $d = [K : \mathbb{Q}]$ and $\mathfrak{o}_K$ be its ring of integers. A free $\mathfrak{o}_K$-submodule L of rank $n + 2$ of $\mathbb{R}^{n+1,1}$ is called a $\mathfrak{o}_K$-lattice. Let $(e_1, \ldots, e_{n+1})$ be a basis of L. Assume that the entries of its Gram matrix $G = ((e_i, e_j)) = (g_{ij})$ belong to $\mathfrak{o}_K$ Let $\sigma_i : K \hookrightarrow \mathbb{R}, i = 1, \ldots, d-1$, be the set of non-identical embeddings of K into $\mathbb{R}$. We assume additionally that the matrices $G^{\sigma_i} = (\sigma_i(g_{ij}))$ are positive definite. Let

$$f = \sum_{1 \le i,j \le n+2} g_{ij}t_it_j$$

be the quadratic form defined by the matrix G and let $\mathrm{O}(f, \mathfrak{o}_K)$ be the subgroup of $\mathrm{GL}_{n+1}(\mathfrak{o}_K)$ of transformations that leave f invariant. The group $\mathrm{O}(f, \mathfrak{o}_K)$ is a discrete subgroup of $\mathrm{O}(\mathbb{R}^{n+1,1})$ with a fundamental polyhedron of finite volume, compact if f does not represent zero. By passing to the projective orthogonal group, we will view such groups as Kleinian subgroups of $\mathrm{Iso}(\mathbb{H}^{n+1})$. A subgroup of $\mathrm{Iso}(\mathbb{H}^{n+1})$ which is commensurable with a subgroup of the form $\mathrm{PO}(f, \mathfrak{o}_K)$ is called *arithmetic*. Two groups $\mathrm{PO}(f', \mathfrak{o}'_K)$ and $\mathrm{PO}(f, \mathfrak{o}_K)$ are commensurable if and only if $K = K'$ and f is equivalent to $\lambda f'$ for some positive $\lambda \in K$. In particular, any subgroup of finite index in $\mathrm{PO}(f, \mathfrak{o}_K)$ is an arithmetic group. A Bianchi group $\mathrm{Bi}(m)$ which we considered before are examples of arithmetic Kleinian groups.

We will be interested in the special case of an *integral lattice* where $K = \mathbb{Q}$. A Kleinian group Γ in $\mathbb{H}^{n+1}$ is called *integral* if it is commensurable with a subgroup of $\mathrm{PO}(f, \mathbb{Z})$ for some integral quadratic form f of signature $(n + 1, 1)$. We will be dealing mostly with geometrically finite Kleinian groups with infinite covolume. However, it follows from [6], Proposition 1, that such a group is always Zariski dense in $\mathrm{PSO}(n + 1, 1)$ or $\mathrm{PO}(n + 1, 1)$ if it acts irreducibly in $\mathbb{H}^{n+1}$. In terminology due to P. Sarnak, Γ is a *thin group*.

Remark 7.6. Let Γ be an integral Kleinian group of isometries of $\mathbb{H}^n$ of finite covolume, for example, the orthogonal group of some integral quadratic lattice L. It is obviously geometrically finite, however, for $n > 3$ it may contain finitely generated subgroups which are not geometrically finite. In fact, for any lattice L of rank ≥ 5 that contains a primitive sublattice $I^{1,3}$, the orthogonal group $\mathrm{O}(L)$ contains finitely generated but not finitely presented subgroups (see [31]).

This shows that not every group of automorphisms of an Enriques or a K3 surface is geometrically finite. For example, we can consider the lattice $M = I^{1,4}(2) = \langle 2 \rangle \perp A_1^{\oplus 4}$.[2] Its group of automorphisms is isomorphic to $\mathrm{O}(I^{1,4})$ and hence it is not finitely presented as a group of isometries of the hyperbolic space $\mathbb{H}^9$. It is easy to embed M in the lattice E_{10} isomorphic to the numerical lattice of an Enriques surface. We map a copy of $\le 2\rangle$ to $\mathbb{Z}(f + g)$, where (f, g) are the standard generators of the hyperbolic summand U. One copy of A_1 we map to $\mathbb{Z}(f - g)$. The other copies

[2] Here, for any quadratic lattice M, we denote by $M(k)$ the same abelian group M but the quadratic form multiplied by m.

of A_1 we map to three orthogonal sublattices of the summand E_8. A subgroup of finite index of $\mathrm{O}(M)$ extends to a subgroup of $\mathrm{O}(\mathsf{E}_{10})$ that acts identically on the orthogonal complement. Taking X general enough, and using that $\mathrm{Aut}(X)^*$ is of finite index in $\mathrm{O}(N^1(X))$, we realize a finite index subgroup of $\mathrm{O}(M)$ as a group of automorphisms of X.

A similar construction works for K3 surfaces. This time we take the lattice $I^{1,4}(4)$ in order to get a K3 surface without smooth rational curves. By taking X general enough, we see that a subgroup of finite index of $\mathrm{O}(M)$ is realized as a group of automorphisms of the surface. This shows that there exist K3 surfaces which admit automorphism groups which are not geometrically finite.

Note that the full automorphism group of a K3 surface is geometrically finite. In fact, it is known that the G group generated by automorphism groups of X and reflections in the divisor classes of smooth rational curves is known to be of finite index in the orthogonal group of $\mathrm{Num}(X)$. If we write the fundamental domain of the group G in $\mathbb{H}^n$, we find a finite polyhedron with walls H_δ corresponding to vectors δ with $(\delta, \delta) = -2$ and the other walls. The group of automorphisms has a finite polyhedral domain in the polytope bounded by the walls H_δ. If drop the walls H_δ, we remaining walls will define a finite polyhedral domain for a group which contains $\mathrm{Aut}(X)$ as a subgroup of finite index.[3]

A sphere packing $\mathfrak{P}(P)$ in $\hat{E}^n$ defined by a Boyd-Maxwell polytope P is called *integral* if P is a lattice polytope and the there exists a real number λ such that the curvature k of any sphere in $\mathfrak{P}$ belongs to $\lambda\mathbb{Z}$. The number λ is called the *scale* of the integral packing.

Let $N(P)$ be the $\mathbb{Z}$-span of the vectors

$$v_i = \omega_i / c_i.$$

Since

$$c_i \mathfrak{e}_i = \sum_{k=1}^{n+1} g_{ik} c_i \omega_k = \sum_{k=1}^{n+1} c_i c_k g_{ik} \omega_k,$$

we obtain that $M(P) \subset N(P)$. Let

$$C = \mathrm{diag}(c_1, \ldots, c_{n+1}),$$

Then $CG(P)C$ is an integral matrix, hence $dC^{-1}G(P)^{-1}C^{-1}$ is an integral matrix, where

$$d = |C|^2 |G(P)|.$$

This implies that

$$b_{ij} = dg^{ij}/c_i c_j \in \mathbb{Z},$$

and hence

$$dv_i = \sum_{k=1}^{n+1} g^{ik} \mathfrak{e}_k / c_i = \sum_{k=1}^{n+1} b_{ik} c_k \mathfrak{e}_k$$

[3]This argument wasd explained to me by V. Nikulin and I. Shimada.

This shows that $dN(P) \subset M(P)$, and hence $N(P) \subset M(P) \otimes \mathbb{Q}$. We also $(v_i, c_j\mathfrak{e}_j) = \delta_{ij}$, hence $N(P)$ is the dual lattice of $M(P)$ and the discriminant of $M(P)$ is a divisor of d.

Theorem 7.7. *Let P be a lattice Boyd-Maxwell polytope with multiplier $(c_1, \ldots, c_{n+1})$ and Gram matrix $G(P) = (g_{ij})$. Suppose that $c_i/\sqrt{-g^{ii}}$, where $g^{ii} < 0$, does not depend on i and $M(P)$ contains an isotropic vector. Then there exists an isometry σ of $\mathbb{H}^n$ such that the image of $\sigma(P)$ defines an integral sphere packing with the scale $c_i/\sqrt{-g^{ii}}$.*

Proof. Choose a $\mathbb{Z}$-basis $v_1, \ldots, v_{n+1}$ in $M(P)$ with Gram matrix $C\cdot G(P)\cdot C$. Let $f = \sum_{i=1}^{n+1} a_i v_i$ be an isotropic vector in $M(P)$. We can find vectors $w_2, \ldots, w_{n+1} \in \mathbb{R}^{n+1}$ such that the matrix X whose first rows is the vector $(a_1, \ldots, a_{n+1})$ and the remaining n rows are the vectors w_i satisfies

$$X \cdot (C \cdot G(P) \cdot C) \cdot {}^tX = (X \cdot C) \cdot G(P) \cdot {}^t(X \cdot C) = J_{n+1}.$$

As in the proof of Theorem 7.3, we deduce from this that

$$G(P)^{-1} = {}^t(X \cdot C) \cdot J_{n+1} \cdot (X \cdot C).$$

This shows that the columns of ${}^t(X \cdot C)^{-1}$ can be taken as the normal vectors $\mathbf{e}_i'$ of a polytope P' with $G(P) = G(P')$. Since $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1})$ and $(\mathfrak{e}_1', \ldots, \mathfrak{e}_{n+1}')$ are two bases of the same space $\mathbb{R}^{1,n}$ with the same Gram matrix, there exists an isometry of $\mathbb{R}^{1,n}$ that sends one basis to another. It defines an isometry σ of $\mathbb{H}^n$ such that $\sigma(P) = P'$. The columns of $X \cdot C$ are the fundamental weights $\omega_1', \ldots, \omega_{n+1}'$ of P'. The first coordinates of these vectors are equal to $(c_1 a_1, \ldots, c_{n+1}a_{n+1})$. Suppose $g^{ii} < 0$, i.e. ω_i is a real fundamental weight. Then $k_i = c_i a_i/\sqrt{-g^{ii}}$ is the curvature of the sphere defined by the normalized weight $\bar{\omega}_i$.

From now on, to simplify the notation, we assume that $P = P'$. Since the vectors $v_i = \omega_i/c_i = \sqrt{-g^{ii}}\bar{\omega}_i/c_i$ span the Γ_P-invariant lattice $N(P)$, we obtain that any vector $\gamma(\bar{\omega}_i), \gamma \in \Gamma_P$, is equal to a linear combination of the vectors $c_i\sqrt{-g^{ii}}v_i$ with integral coefficients. This shows that the curvature of the sphere defined by $s(\bar{\omega}_i)$ is equal to a linear combination of the numbers $c_i/\sqrt{-g^{ii}}a_i$ with integral coefficients, and hence is an integer. □

Remark 7.8. The condition that $M(P)$ contains an isotropic vector is always satisfied is $\mathrm{rank}\ M(P) = n + 1 \geq 5$. Also, if there is only one real fundamental root, any sphere packing defined by a lattice Boyd-Maxwell polytope is integral.

Let us see how to compute g^{ii} if we know the Coxeter diagram of P. It is clear that g^{ii} is equal to $\Delta_{ii}/|G(P)|$, where Δ_{ii} is the principal $n \times n$-minor of $G(P)$ obtained by deleting the i-column and the ith row. It is equal to the determinant of the Gram matrix of $\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1}$ with $\mathfrak{e}_i$ omitted.

Example 7.9. Lattice Boyd-Maxwell polytopes exist in all dimensions $n \leq 10$. In dimension $n \geq 7$ they are all lattice polytopes. Here are examples of lattice Boyd-Maxwell polytopes in $\mathbb{H}^7$: Here $+$ means that the vertex corresponds to a real fundamental weight. The multiplier vector for each t

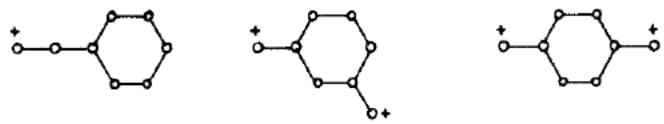


Figure 7.4: Examples of lattice Boyd-Maxwell polytopes in $\mathbb{H}^7$

polytope is $\sqrt{2}(\sqrt{2}, 1, 1, 1, 1, 1, 1, 1)$. The matrix

$$C \cdot G(P) \cdot C = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

One can show that

$$M(P) \cong \mathsf{E}_6 \perp \mathsf{U}(2). \tag{7.15}$$

The discriminant group $M(P)^\vee/M(P)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and the discriminant is equal to -12.

We have $g^{ii} < 0$ only the vertices which come with sign $+$. An easy computation gives that $\sqrt{-g^{ii}} = 1/\sqrt{6})$ and $c_i = \sqrt{2}$. So the packing is integral with the scale equal to $\sqrt{12}$.

In the second example, the multiplier is $\sqrt{2}(1, \ldots, 1)$. The matrix

$$C \cdot G(P) \cdot C = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

One can show that

$$M(P) \cong \mathsf{A}_5 \perp \mathsf{A}_1 \perp \mathsf{U}.$$

The discriminant group $M(P)^\vee/M(P)$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$ and the discriminant is equal to -12.

We have $g^{ii} < 0$ only for the first extreme vertex on the left. An easy computation gives that $\sqrt{-g^{ii}} = 1$ and $c_i = \sqrt{2}$. So the packing is integral with the scale equal to $\sqrt{2}$.

Example 7.10. Let P be an Apollonian polyhedron in $\mathbb{H}^3$. We know that $G(P) = \mathrm{circ}(-1,1,1,1)$, so it is an integral matrix. The lattice $M(P)$ is an odd lattice. In the basis formed by the vectors $\mathfrak{e}_1, \mathfrak{e}_1+\mathfrak{e}_2, \mathfrak{e}_1+\mathfrak{e}_3, \mathfrak{e}_1+\mathfrak{e}_2+\mathfrak{e}_3-\mathfrak{e}_4$, the Gram matrix becomes equal to

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

This shows that,

$$M(P) \cong \langle -1 \rangle \oplus \mathsf{U}(2) \oplus \mathsf{A}_1(2). \tag{7.16}$$

For applications to algebraic geometry, we need to consider the maximal even sublattice $M(P)^{\mathrm{ev}}$ of $M(P)$. It is easy to see that

$$M(P)^{\mathrm{ev}} \cong \mathsf{U}(2) \perp \mathsf{A}_1(2) \perp \mathsf{A}_1(2). \tag{7.17}$$

Since the entries of $G(P)$ are integers, the multiplier is the vector $(1,1,1,1)$. Also $g^{ii} = -1/4$ for all i. This shows that P is a lattice polytope that defines an integral packing with the scale equal to 1.

Remark 7.11. Suppose we have an integral sphere packing defined by a lattice Boyd-Maxwell polytope. Thus the curvatures k_S of its spheres S are equal to λm_S, where m_S are integers. One may ask whether the density of integers m_S is positive in the set of all integers. In the case of the Apollonian circle packing the answer is positive. This is a recent result of J. Bourgain and E. Fuchs [9].

Finally, let us see that, for any lattice Boyd-Maxwell polytope, the group Γ_P is realized as the group $\mathrm{Aut}(X)^*$ for some K3 surface.

In order to do it, we first realize the lattice $M(P)$ as the Picard lattice $N^1(X)$ for some K3 surface. We use the following proposition from [41], Theorem 1.14.4.

Proposition 7.12. *Let M be an even quadratic lattice of signature $(1,n)$. Suppose $n \le 18$, and the minimal number of generators of the discriminant group is less than $20-n$. then there exists unique primitive embedding of M in the lattice* $\mathsf{U} \perp \mathsf{U} \perp \mathsf{U} \perp \mathsf{E}_8 \perp E_8$.

Applying the theory of periods, we deduce from this proposition that there exists a $19-n$-dimensional family of non-isomorphic algebraic K3 surfaces such that $M \subset N^1(X)$. For a general member of this family, we have the equality. We know that $\mathrm{rank}\ M(P) \le 11$, i.e. $n \le 10$. Suppose $n \le 9$. Then the number of generators of the discriminant group is less than or equal than the rank. This implies that the lattice $M(P)(2)$ satisfies the conditions of the proposition. Hence there exists a K3 surface with $N^1(X) \cong M(P)(2)$. Since $M(P)$ does not have vectors of norm -2, the surface does not contain smooth rational curves. Applying Theorem 6.4, we obtain that

$$\Gamma_P \subset \mathrm{Aut}(X)^*.$$

Figure 7.5: Boyd-Maxwell polytopes in $\mathbb{H}^{10}$

The only remaining case to consider is when $n = 10$. There are twoe lattice Boyd-Maxwell polytopes in $\mathbb{H}^{10}$. Its Coxeter diagrams are the following.

The first one has multiplier vector equal to $(1, \sqrt{2}, \ldots, \sqrt{2})$. The matrix $C \cdot G(P) \cdot C$ is equal to the matrix

$$\begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & a_{11} & \ldots & a_{19} \\ 0 & 0 & a_{21} & \ldots & a_{29} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{91} & \ldots & a_{99} \end{pmatrix},$$

where (a_{ij}) is the matrix defining the lattice E_9. The determinant of this matrix is equal to 1. Since the lattice M defined by this matrix is odd, we obtain that it is isomorphic to $I^{1,10}$. The principal minor Δ_{11} is equal to the discriminant of the lattice E_{10}, hence equal to 1. The orthogonal complement of the sublattice E_{10} defined by this matrix is of discriminant -1. Thus we obtain that

$$M(P) \cong \langle -1 \rangle \perp \mathsf{E}_{10},$$

hence

$$M(P)^{|ev} \cong \mathsf{A}_1 \oplus \mathsf{E}_{10} \cong \mathsf{E}_{11}.$$

Obviously, for the second polytope P', we have $M(P')$ is the same lattice as $M(P)^{\text{ev}}$. In both cases

$$M(P)^{\text{ev}}(2) \cong M(P)(2) \cong \mathsf{E}_{11}(2) \cong \mathsf{U}(2) \perp \mathsf{E}_8(2) \perp \mathsf{A}_1(2).$$

This case still covered by Theorem 1.14.4 that treats also the case when the minimal number of generators of the discriminant group is equal to the rank (the assumption from the theorem is satisfied if $\mathsf{U}(2)$ is a direct summand of M). Thus, we can realize $M(P)^{\text{ev}}(2)$ and $M(P)$ as the Picard lattice of a K3 surface, and obtain the following theorem.

Theorem 7.13. *Let P be a lattice Boyd-Maxwell polytope in $\mathbb{H}^n$. Then there exists a K3-surface X (depending on $19 - n$ parameters) such that*

$$\Gamma_P \subset \mathrm{Aut}(X)^*.$$

Note that Γ_P is a subgroup of the reflection group of the lattice $M(P)(2)$. The lattice $M(P)(2)$ is reflexive if and only if the lattice $M(P)$ is reflexive. For most of the Boyd-Maxwell polytopes the lattice $M(P)$ nor $M(P)^{\text{ev}}$ is reflexive. For example, this is true for the Apollonian lattice. So, the group Γ_P is of infinite index in $\mathrm{Aut}(X)^*$.

Lecture 8

Orbital degree counting

Let X be a smooth projective algebraic surface with a fixed projective embedding $X \hookrightarrow \mathbb{P}^N$ that defines the class $h \in N^1(X)$ of a hyperplane section. Let $c = [C] \in N^1(X)$ be the numerical class of an irreducible curve C on X and $\mathsf{O}_\Gamma(c)$ be its orbit with respect to a subgroup Γ of $\mathrm{Aut}(X)^*$. Note that $h \cdot c$ is equal to the degree of C in $\mathbb{P}^N$.

For any real $T > 0$, let

$$\mathsf{N}_{c,h,\Gamma}(T) = \#\{c' \in \mathsf{O}_\Gamma(c) : (h, c) < T\}.$$

We would like to find an asymptotic of this function when T goes to infinity. It turns out that it can be expressed in terms of the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of the group Γ when Γ is not an elementary discrete group.

Let A be a subset of the Euclidean space $\mathbb{R}^n$. Recall that the *Hausdorff dimension* $\mathrm{H.dim}(A)$ of A is defined to be the infimum for all $s \geq 0$ for which

$$\mu_s(A) = \inf_{A \subset \cup_j B_j} (\sum_j r(B_j)^s) = 0. \tag{8.1}$$

Here (B_j) is a countable set of open balls of radii $r(B_j)$ which cover A.

For example, if the Lebesque measure of A is equal to 0, then (8.1) holds for $s = n$, hence $\mathrm{H.dim}(A) \leq n$. The Hausdorff dimension coincides with the Lebesque measure if the latter is positive and finite. A countable set has the Hausdorff measure equal to zero. Also it is known that the topological dimension of A is less than or equal to its Hausdorff dimension.

The Hausdorff dimension is closely related to the *fractal dimension* of a *fractal set* A, i.e. a set that can be subdivided in some finite number $N(\lambda)$ of subsets, all congruent (by translation or rotation) to one another and each equal to a scaled copy of A by a linear factor λ. It is defined to be equal to $\frac{\log N(s)}{\log(1/\lambda)}$. For example, the Cantor set consists of two parts A_1 and A_2 (contained in the interval $[0, 1/3]$ and $[1/3, 1]$), each rescaled version of the set with the scaling factor $1/3$. Thus its fractal dimension is equal to $\log 2/\log 3 < 1$. It is clear that the Hausdorff dimension of a bounded fractal set of diameter $D = 2R$ is less than or equal than the fractal dimension. In fact, such a set can be covered by $N(\lambda)$ balls of radius λR, or by $N(\lambda)^2$ balls of radius $\lambda^2 R$. Since $\lambda < 1$, we get $\mu_s(A) = \lim_{k\to\infty} \frac{N(\lambda)^k}{(\lambda^k R)^s}$ which is zero if $N(\lambda)\lambda^s < 1$ or $s > \log N/\log(1/\lambda)$. In fact, the Hausdorff dimension of the Cantor set coincides with its fractal dimension $\log 2/\log 3$.

By a theorem of D. Sullivan [50], for a geometrically finite non-elementary discrete group Γ, the Hausdorff dimension δ_Γ of $\Lambda(\Gamma)$ is positive and coincides with the *critical exponent* of Γ equal to

$$\inf\{s > 0 : \sum_{g\in\Gamma} e^{-sd(x_0,g(x_0))} < \infty\}, \tag{8.2}$$

where x_0 is any point on $\mathbb{H}^n$. Using this equality, Sullivan shows that

$$\delta_\Gamma = \overline{\lim_{T\to\infty}} \frac{\log N_T}{R}, \tag{8.3}$$

where N_T is the number of orbit points y with hyperbolic distance from x_0 less than or equal than R. He further shows in [51], Corollary 10, that under the additional assumption that Γ has no parabolic fixed points, that there exists constants c, C such that, as $T \to \infty$,

$$ce^{T\delta_\Gamma} \le N_T \le Ce^{T\delta_\Gamma}. \tag{8.4}$$

In particular, asymptotically, as $T \to \infty$,

$$N_T \sim c(T)e^{T\delta_\Gamma},$$

where $\lim\limits_{T\to\infty} \frac{c(T)}{T} = 0$.

If $\delta_\Gamma > \frac{1}{2}(n-1)$, then P. Lax and R. Phillips show that, for any geometrically finite non-elementary discrete group Γ, the function $c(T)$ is a constant depending only on Γ. When Γ is of finite covolume, then δ_Γ is known to be equal to $n-1$, and the result goes back to A. Selberg. The assumption on δ_Γ has been lifted by T. Roblin [46].

Recall that, in the vector model of $\mathbb{H}^n$, the hyperbolic distance $d(x, y)$ between two points is expressed by (2.1) in terms of the inner product (x, y), where x, y are represented by vectors in $\mathbb{R}^{1,n}$ with norm equal to 1. Taking $x = [h]$ and $y = [c]$, where $C^2 > 0$, we obtain that

$$\mathsf{N}_T(c, h, \Gamma) \sim c_{\Gamma,c,h}T^{\delta_\Gamma},$$

for some constant $c_{\Gamma,c,h}$.

In order to obtain similar formula in the cases when $C^2 \le 0$, we need to replace the family of hyperbolic balls with another family of sets of growing volume. In Lecture 3, we defined the distance form a point $x \in \mathbb{H}^n$ to any point $[e] \in \mathbb{P}^n(\mathbb{R})$.

In each of the three cases" $[e] \in \mathbb{H}^n$, $[e] \in \partial\mathbb{H}^n$, $[e] \notin \overline{\mathbb{H}}^n$, let us consider the sets

$$B_T(e) = \{x \in \mathbb{H}^n : d(x, [e]) \le T\}.$$

Let Γ be a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$ and let $x_0 \in \mathbb{H}^n$. We have the following result which is combined effort of several people ([44], Theorem 1.2 and [40], Corollary 7.14), see a nice survey of some of these results in [43]).

Theorem 8.1. *Let Γ be a geometrically finite non-elementary discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$. Let $[e] \in \mathbb{P}^n(\mathbb{R})$. If $(e,e) \le 0$, assume that $\Gamma^+ = \Gamma \cap \mathrm{Iso}(\mathbb{H}^n)^+$ is Zariski dense in the Lie group $\mathrm{Iso}(\mathbb{H}^n)^+$ and also that $\delta_\Gamma > 1$ if $(e,e) < 0$. Then*

$$\lim_{T\to\infty} \frac{\#\{\mathsf{O}_\Gamma(x_0) \cap B_T(e)\} \le T}{T^{\delta_\Gamma}} = c_{\Gamma,x_0,[e]},$$

where $c_{\Gamma,x_0,[e]}$ is a positive constant that depends only on $\Gamma, x_0, [e]$.

Note that the results from [40] also give error terms.

Remark 8.2. In fact, the assertion which we use is stated in other terms, however, using Proposition 3.1, one can show that this is equivalent to the assertion of the theorem.

Remark 8.3. Note that the condition that Γ^+ is Zariski dense in $\mathrm{Iso}(\mathbb{H}^n)^+$ is satisfied if Γ originates from an irreducible representation in $\mathbb{R}^{1,n}$ ([6], Proposition 1).

In the case when $(e,e) > 0$ (resp. $(e,e) = 0$ and $n = 3$), Theorem 8.1 follows from [44], Theorem 1.2 (resp. [33], Theorem 2.10) that gives an asymptotic of the number of orbit points in a ball $\{[v] \in \mathbb{H}^n : ||v|| \le T\}$, where $||x||$ is the Euclidean norm. One has only use that if $v = (x_0, \ldots, x_n)$ with $\sum_{i=0}^n x_i^2 \le T^2$ and $x_0^2 - \sum_{i=1}^n x_i^2 = 1$ (resp. $= 0$), then $(v,(1,\ldots,0))^2 = x_0^2 \le (T^2+1)/2$ (resp. $\le T^2$).

In the case $(e,e) > 0$, and Γ and $\Gamma \cap G_{[e]}$ are of finite covolume (the second condition is always satisfied if $n \ge 3$ [15]), Theorem (8.1) follows from [23], [24]. In this case $\delta_\Gamma = n - 1$.

Using that $d(\gamma(g^{-1}([e]), x_0) = d([e], \gamma(x_0))$, and taking $x_0 = [h], [e] = [C]$, we obtain

Corollary 8.4. *Let X be a surface of Kodaira dimension 0 or a rational surface obtained by blowing up $N \ge 10$ points in $\mathbb{P}^2$. Let $\Gamma \subset \mathrm{Aut}(X)^*$ be a non-elementary subgroup. Let h be the numerical class of a hyperplane section of X in some projective embedding and $c = [C]$ be the numerical class of some irreducible curve on X. If $C^2 \le 0$, assume that Γ acts irreducibly in $N^1(X)_\mathbb{R}$. Then*

$$\lim_{T\to\infty} \frac{\mathsf{N}_{\Gamma,C,h}}{T^{\delta_\Gamma}} = c_{\Gamma,x_0,[e]},$$

where $c_{\Gamma,c,h}$ is a positive constant that depends only on Γ, c, h.

Example 8.5. Let $\Gamma = \mathsf{UC}(4)$ be the universal Coxeter group. We can realize it as a subgroup of $\mathrm{Aut}(X)^*$, where X is a K3 surface with the Picard lattice given in (7.17). However, I do not know an explicit construction of a K3 surface whose Picard lattice is isomorphic to this lattice. Instead we consider a surface X in $(\mathbb{P}^1)^4$ from example 6.3. The Picard lattice of this surface is given by the matrix $\mathrm{circ}(0,2,2,2)$. If we change the basis (f_1, f_2, f_3, f_4) to $(f_1, f_2, f_3 - f_1 + f_2, f_1 + f_2 - f_4)$, then the matrix becomes equal to $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}$. This shows that

$$N^1(X) \cong \mathsf{U}(2) \perp \mathsf{A}_2(2).$$

The limit set of $\mathsf{UC}(4)$ has been computed up to nine decimals in [52] (see also [37]). We have

$$\delta_{\mathsf{UC}(4)} = 1.305686729....$$

Since $\mathsf{UC}(4)$ acts irreducibly in $\mathbb{H}^3$, we can apply Corollary 8.4 to obtain the asymptotic of the growth of degrees of curves on X in an orbit of Γ.

One can also realize $\mathsf{UC}(4)$ as acting on an Enriques surface. We consider the lattice

$$M(P)^{\mathrm{ev}}(1/2) \cong \mathsf{U} \perp \mathsf{A}_1$$

and embed it primitively in the numerical lattice $N^1(X)$ of an Enriques surface. Recall that the latter is isomorphic to $\mathsf{E}_{10} \cong \mathsf{U} \perp \mathsf{E}_8$. We assume that X has no smooth rational curves. So, the automorphism group of X is a subgroup of finite index in $\mathrm{O}(\mathsf{E}_{10})$. This shows that a subgroup of finite index of $\mathsf{UC}(4)$ is realized as a group of automorphisms of X.

Example 8.6. Consider the Coxeter group $\Gamma(a,b,c)$ of a Coxeter triangle $\Delta(a,b,c)$ in $\mathbb{H}^2$ defined by the Gram matrix $\begin{pmatrix} -1 & a & b \\ a & -1 & c \\ b & c & -1 \end{pmatrix}$, where a, b, c are rational numbers ≥ 1. If $a = b = c = 1$, the fundamental domain is an ideal triangle. If $a, b, c > 1$, the fundamental triangle is as on the following picture.

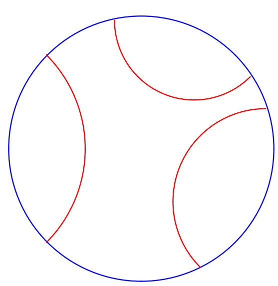


Figure 8.1: Hyperbolic triangle with no vertices

Obviously the triangles are lattice Coxeter polytopes with multiplier $(\sqrt{N}, \sqrt{N}, \sqrt{N})$, where N is the least common denominator of a, b, c. We know that $\Gamma(a,b,c)$ leaves the lattice M generated by $N\mathfrak{e}_i$'s invariant. Since the rank of M is small, the lattice $M(2)$ does not contain vectors of norm -2 and can be realized as the Picard lattice of some K3 surface X without smooth rational curves. Thus $\Gamma(a,b,c)$ can be realized as a subgroup of $\mathrm{Aut}(X)^*$. For example, let us consider the case

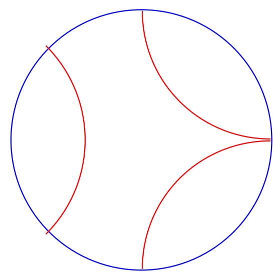


Figure 8.2: Hyperbolic triangle with one ideal point

$(a, b, c) = (a, a, 1)$. The fundamental triangle has one ideal point and looks like in the following figure 8.6.

Note that when $r = a = 1$, the lattice L_r is isomorphic to the Apollonian lattice $\mathcal{A}p_1$ and the group Γ_1 is isomorphic to the Apollonian group Ap_1.

Let $\mathcal{H}^2 \to \mathbb{H}^2$ be the map from the upper-half plane to the unit disk given by the map $z \mapsto \frac{z-i}{z+i}$. One can show that the pre-image of the sides of our triangle are the lines $x = 1, x = -1$ and the upper half-circle of radius $r = 1/a$ with center at the origin. Recall that $a \leq -1$ so that the half-circle is between the vertical lines. Let us re-denote our group $\Gamma_{a,a,1}$ by Γ_r. It was shown by C. McMullen in [37] that as $r \to 0$, we have

$$\delta_{\Gamma_r} = \frac{r+1}{2} + O(r^2),$$

while for $r \to 1$, we have

$$\delta_{\Gamma_r} \sim 1 - \sqrt{1-r}.$$

Example 8.7. Let us give another example of a realizable group $\Gamma(a, b, c)$. It is taken from [3]. Let X be a K3 surface defined over an algebraically closed field of characteristic $\neq 2$ embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ as a complete intersection of hypersurfaces of multi-degree $(1, 1)$ and $(2, 2)$. Let $p_1, p_2 : X \to \mathbb{P}^2$ be the two projections. They are morphisms of degree 2 branched along a plane curve B_i of degree 6. We assume that B_1 is nonsingular and B_2 has a unique double point q_0 so that the fiber $p_2^{-1}(q_0)$ is a smooth rational curve R that is mapped isomorphically under p_1 to a line. We assume that X is general with these properties. More precisely, we assume that $\mathrm{Pic}(X)$ has a basis (h_1, h_2, r), where $h_i = p_i^*(\text{line})$ and r is the class of R. The intersection matrix of this basis is equal to

$$\begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Let $s = p_1^*((p_1)_*(r)) - r = h_1 - r$. It is the class of smooth rational curve S on X. The pre-image of the pencil of lines through q_0 is an elliptic pencil $|F|$ on X with $[F] = h_2 - r$. The curve S is a section of the elliptic fibration defined by the linear system $|F|$ and the curve R is its 2-section that intersects S with multiplicity 3. Consider the following three automorphisms of X. The first two Φ_1 and Φ_2 are defined by the birational deck transformations of the covers p_1 and p_2. The third one Φ_3 is defined by the negation automorphism of the elliptic pencil with the group law defined by the choice of S as the zero section.

It is easy to compute the matrix of each Φ_i in the basis $(f, s, r) = ([F], [S], [R])$ with the Gram matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -2 \end{pmatrix}.$$

We have $\Phi_1^*(s) = r, \Phi_1^*(r) = s$ and $f' = \Phi_1^*(f) = af + bs + cr$. Since Φ_i^2 is the identity and $(f, f) = 0$, we get $a = -1$ and $b = c$. Since $(f', s) = (f, r) = 2$, we easily get $b = c = 3$. The matrix of Φ_1, and similarly obtained matrices of Φ_2 and Φ_3 are as follows.

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 14 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

The transformations $\Phi_1' = \Phi_1 \circ \Phi_2 \circ \Phi_1, \Phi_2, \Phi_3$ are the reflections with respect to the vector α_i, where

$$\alpha_1 = -4f + 13s + 10r, \ \alpha_2 = 4f - 2s + r, \ \alpha_3 = 7f + 2s - r.$$

The Gram matrix of the vectors $\alpha_1, \alpha_2, \alpha_3$ is equal to

$$G = \begin{pmatrix} -22 & 143 & 220 \\ 143 & -22 & 22 \\ 220 & 22 & -22 \end{pmatrix} = -22 \begin{pmatrix} 1 & -\frac{13}{2} & -10 \\ -\frac{13}{2} & 1 & -1 \\ -10 & -1 & 1 \end{pmatrix}.$$

So, the group generated by Φ_1', Φ_2, Φ_3 coincides with the triangle group $\Gamma(\frac{13}{2}, 10, 1)$. The fundamental triangle P has one ideal vertex. The reflection group Γ_P is a subgroup of infinite index of the group Γ of automorphisms of X generated by Φ_1, Φ_2, Φ_3. Baragar proves that Γ is isomorphic to $\mathrm{Aut}(X)$ (for sufficiently general X). He finds the following bounds for δ_Γ

$$.6515 < \delta_\Gamma < .6538.$$

Example 8.8. This is again due to Baragar [4]. We consider a nonsingular hypersurface X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2, 2, 2)$. It is a K3 surface whose Picard lattice contains the Apollonian lattice

$\mathcal{A}p(1)$. If X is general, then the Picard lattice coincides with this lattice. We assume that one of the projections $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$, say p_{12}, contains the whole $\mathbb{P}^1$ as its fiber over some point $q_0 \in \mathbb{P}^1 \times \mathbb{P}^1$. All the projections are degree 2 maps. Let $F_i, i = 1, 2, 3$, be the general fibers of the projections $p_i : X \to \mathbb{P}^1$. Each F_i is an elliptic curve whose image under the map p_{jk} is a divisor of type $(2, 2)$. Let f_1, f_2, f_3, r be the classes of the curves F_1, F_2, F_3, R. We assume that X is general with these properties so that $\mathrm{Pic}(X)$ is generated by these classes. The Gram matrix of this basis is equal to

$$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to see that

$$\mathrm{Pic}(X) \cong \mathsf{U} \oplus \begin{pmatrix} -4 & 2 \\ -2 & -8 \end{pmatrix}.$$

According to Vinberg's classification of 2-reflective hyperbolic lattices of rank 4 [55], the Picard lattice is not 2-reflective. Hence the image of the group $\mathrm{Aut}(X)$ in $\mathrm{O}(\mathrm{Pic}(X))$ is of infinite index.

Let Φ_{ij} be the automorphisms of X defined by the deck transformations of the projections p_{ij}. Let Φ_4' be defined as the transformation Φ_3 in the previous example with respect to the elliptic pencil $|F_3|$ with section R. The transformation Φ_{12}^* leaves the vectors f_1, f_2, r invariant, and transforms f_3 to $2f_1+2f_2-f_3-r$. Thus Φ_{12}^* is the reflection with respect to the vector $\alpha_1 = -2f_1-2f_2+2f_3+r$.

The transformation Φ_{13}^* leaves f_1, f_3 invariant and transforms r in $r' = f_1 - r$. It also transforms f_2 to some vector $f_2' = af_1 + bf_2 + cf_3 + dr$. Computing $(f_2', f_1) = (f_2, f_1), (f_2', f_3) = (f_2, f_3), (f_2', r) = (f_2, f_1 - r)$, we find that $f_2' = 2f_1 - f_2 + 2f_3$. Similarly, we find that $\Phi_{23}^*(f_2) = f_2, \Phi_{23}^*(f_3) = f_3, \Phi_{23}^*(r) = f_2 - r$ and $\Phi_{23}^*(f_2) = -f_1 + 2f_2 + 2f_3$.

It follows from the definition of a group law on an elliptic curve that

$$\Phi_4'^*(f_3) = f_3,\ \Phi_4'^*(r) = r,\ \Phi_4'^*(f_i) = -f_i + 8f_3 + 4r,\ i = 1, 2$$

Consider the transformations

$$\Phi_1 = \Phi_{12},\ \Phi_2 = \Phi_{13} \circ \Phi_{12} \circ \Phi_{13},\ \Phi_3 = \Phi_{23} \circ \Phi_{12} \circ \Phi_{23},\ \Phi_4 = \Phi_4' \circ \Phi_{12} \circ \Phi_4'.$$

These transformations act on $\mathrm{Pic}(X)$ as the reflections with respect to the vectors

$$\begin{aligned} \alpha_1 &= -2f_1 - 2f_2 + 2f_3 + r, \\ \alpha_2 &= \Phi_{13}^*(\alpha_1) = -5f_1 + 2f_2 - 2f_3 - r, \\ \alpha_3 &= \Phi_{23}^*(\alpha_1) = 2f_1 - 5f_2 - 2f_3 - r, \\ \alpha_4 &= \Phi_4(\alpha_1) = 2f_1 + 2f_2 - 30f_3 - 15r. \end{aligned}$$

The Gram matrix of these four vectors is equal to

$$\begin{pmatrix} -14 & 14 & 14 & 210 \\ 14 & -14 & 84 & 182 \\ 14 & 84 & -14 & 182 \\ 210 & 182 & 182 & -14 \end{pmatrix} = -14 \begin{pmatrix} 1 & -1 & -1 & -15 \\ -1 & 1 & -6 & -13 \\ -1 & -6 & 1 & -13 \\ -15 & -13 & -13 & 1 \end{pmatrix}$$

Let P be the Coxeter polytope defined by this matrix. The Coxeter group Γ_P is generated by the reflections $\Phi_i^*, i = 1, 2, 3, 4$.

Baragar proves that the automorphisms Φ_{ij} and Φ_4' generate a subgroup Γ of $\mathrm{Aut}(X)$ of finite index. His computer experiments suggest that

$$1.286 < \delta_\Gamma < 1.306.$$

Our reflection group Γ_P generated by $\Phi_1, \ldots, \Phi_4$ is of infinite index in Γ. So, we obtain

$$\delta_{\Gamma_P} < 1.306.$$

Lecture 9

Cremona transformations

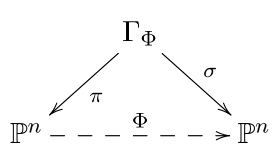
A birational transformation of $\mathbb{P}^n$ over a field $\Bbbk$ is called a *Cremona transformation*. It can be defined, algebraically, as an automorphism of the field of rational functions on $\mathbb{P}^n$ isomorphic to $\Bbbk(z_1, \ldots, z_n)$ or, geometrically, as an invertible rational map given by a formula

$$\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n,\ (t_0, \ldots, t_n) \mapsto (P_0(t), \ldots, P_n(t)),$$

where $P_0, \ldots, p_n$ are mutually coprime homogeneous polynomials of some degree d. The set of indeterminacy points $\mathrm{Bs}(\Phi)$ of Φ is equal to the set of common zeros of the polynomials $P_0, \ldots, P_n$. It has a structure of a closed subscheme of $\mathbb{P}^n$, called the *base scheme* of Φ. All Cremona transformations form a group denoted by $\mathrm{Cr}_n(\Bbbk)$. It is called the *Cremona group* in dimension n. Obviously, $\mathrm{Cr}_n(\Bbbk)$ contains the group of projective transformations of $\mathbb{P}^n$ isomorphic to $\mathrm{PGL}_n(\Bbbk)$. Its elements correspond to transformations defined by linear homogeneous polynomials. Obviously, $\mathrm{Cr}_1(\Bbbk) = \mathrm{PGL}_2(\Bbbk)$, so we assume $n > 1$.

When $n = 2$, according to the famous *Noether Theorem*, the group $\mathrm{Cr}_2(\mathbb{C})$ is generated by $\mathrm{PGL}_2(\mathbb{C})$ and the *standard quadratic transformation* T_2 defined, algebraically, by $(z_1, z_2) \mapsto (1/z_1, 1/z_2)$, and, geometrically, by $(t_0, t_1, t_2) \mapsto (t_1t_2, t_0t_2, t_0t_1)$. It is an involution, i.e. T_2^2 is the identity.

A convenient way to partially describe a Cremona transformation Φ uses the definition of the *characteristic matrix*. As any rational map, Φ defines a regular map Φ_U of an open Zariski subset $U = \mathbb{P}^n \setminus \mathrm{Bs}(\Phi)$ to $\mathbb{P}^n$. Let Γ_Φ denote the Zariski closure of the graph of Φ_U in $\mathbb{P}^n \times \mathbb{P}^n$. Let π and σ be the first and the second projection maps, so that we have the following commutative diagram.



(9.1)

Let $\tilde{\Gamma}_\Phi$ be a resolution of singularities of Γ_Φ, if it exists. If $n = 2$, it always exists and, moreover, we can choose it to be minimal, so that it is uniquely defined, up to isomorphism. It is known that any birational map of nonsingular varieties is a composition of the blow-ups with smooth centers. For any such map $f : X \to Y$ one can see what happens with the Picard group; we have

$$\mathrm{Pic}(X) = f^*(\mathrm{Pic}(Y)) \oplus \mathbb{Z}e,$$

where e is the class in $\mathrm{Pic}(X)$ of the exceptional divisor $f^{-1}(Z)$, wherw Z is the center of the blow-up. This allows one to define two bases in $\mathrm{Pic}(\Gamma_\Phi)$, one comes from π and another one comes from σ. We have $\mathrm{Pic}(X) \cong \mathbb{Z}^r$, and the transition matrix of these two bases is the *characteristic matrix* of T.

Let us consider the case $n = 2$. We have the following factorization of $\pi : X = \tilde{\Gamma}_\Phi \to \mathbb{P}^2$.

$$\pi : X = X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2, \tag{9.2}$$

where $\pi_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. Let

$$E_i = \pi_i^{-1}(x_i), \quad \mathcal{E}_i = (\pi_{i+1} \circ \ldots \pi_N)^{-1}(E_i). \tag{9.3}$$

Let e_i denote the cohomology class $[\mathcal{E}_i]$ of the (possibly reducible) curve $\mathcal{E}_i$. It satisfies $e_i^2 = e_i \cdot K_X = -1$. One easily checks that $e_i \cdot e_j = 0$ if $i \neq j$. Let $e_0 = \pi^*([\ell])$, where ℓ is a line in $\mathbb{P}^2$. We have $e_0 \cdot e_i = 0$ for all i. The classes $e_0, e_1, \ldots, e_N$ form a basis in $N^1(X)$ which we call a *geometric basis*. The Gram matrix of a geometric basis is the diagonal matrix $\mathrm{diag}(1, -1, \ldots, -1)$. Thus the factorization (9.2) defines an isomorphism of quadratic lattices

$$\phi_\pi : I^{1,N} \to N^1(X),\ \mathbf{e}_i \mapsto e_i,$$

where $\mathbf{e}_0, \ldots, \mathbf{e}_N$ is the standard basis of the standard odd unimodular quadratic lattice $I^{1,N}$ of signature $(1, N)$. It follows from the formula for the behavior of the canonical class under a blow-up that K_X is equal to the image of the vector

$$\mathbf{k}_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \ldots + \mathbf{e}_N.$$

This implies that the quadratic lattice $K_X^\perp$ is isomorphic to the orthogonal complement of the vector $\mathbf{k}_N$ in $I^{1,N}$. It is a special case of quadratic lattices considered in Example 5.8. We have

$$\mathbf{k}_N^\perp \cong \mathsf{E}_N$$

The restriction of ϕ_π to $\mathbf{k}_N^\perp$ defines an isomorphism of lattices

$$\phi_\pi : \mathsf{E}_N(-1) \to S_X^0.$$

Let us identify the Coxeter group $W(\mathsf{E}_N)$ with the subgroup of $\mathrm{O}(I^{1,N})$ generated by the reflections in the vectors α_i from Example 5.8. It is contained in the reflection group $\mathrm{Ref}_2(K_X^\perp)$.

The following theorem is due to S. Kantor and goes back to the end of the 19th century. It has been reproved in modern terms by M. Nagata and others.

Theorem 9.1. *Let* $(e_0, \ldots, e_N)$ *be a geometric basis defined by the birational morphism* π. *Then the geometric basis* $(e'_0, e'_1, \ldots, e'_N)$ *is expressed in terms of* $(e_0, e_1, \ldots, e_N)$ *by a matrix* A *which defines an orthogonal transformation of* $I^{1,N}$ *equal to the composition of reflections with respect to the root basis* $(\alpha_0, \ldots, \alpha_{N-1})$ *of* E_N.

The matrix A is the characteristic matrix of Φ. For any plane $H(a) : \sum a_i y_i = 0$ in $\mathbb{P}^s$, its pre-image under the rational map Φ is equal to a curve $V(a) := \sum a_i P_i(t) = 0$ of degree d, where P_i are the polynomials defining Φ. One can show that the class $\pi^{-1}(e'_0) = de_0 - m_1 e_1 - \cdots - m_N e_N$, where m_i are the multiplicities of a general hypersurface $V(a)$ at the points x_i (this has to be carefully defined if the points are infinitely near).

Also, the class $\pi(\sigma^*(e'_i))$ is the class of a curve C_i in $\mathbb{P}^2$ whose image under T is the point y_i defined by e'_i. We have $\sigma^*(e'_i) = d_i e_0 - m_{1i} e_1 - \cdots - m_{Ni} e_N$, where d_i is the degree of the curve C_i and m_{ki} are the multiplicities of C_i at the points x_i. Thus the characteristic matrix A has the following form

$$A = \begin{pmatrix} d & d_1 & \ldots & d_N \\ -m_1 & -m_{11} & \ldots & m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \ldots & -m_{NN} \end{pmatrix}.$$

The matrix A is orthogonal with respect to the inner product given by the matrix J_N. It defines an isometry of the hyperbolic space $\mathbb{H}^N$.

Example 9.2. Let $\Phi = T_2$ be the standard quadratic transformation. Its base scheme consists of three points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]$. The graph of Φ is the closure of points $\{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2 : y = \Phi(x)\}$. In formulas, it is contained in the subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by the condition that

$$\text{rank} \begin{pmatrix} y_0 & y_1 & y_2 \\ t_1 t_2 & t_0 t_1 & t_0 t_1 \end{pmatrix} = 1.$$

Each minor of this matrix is a bihomogeneous polynomial of bidegree $(1, 2)$ in the variables y, t. If $t_0 \neq 0$, we can use the affine coordinates $x_1 = t_1/t_0, x_2 = t_2/t_0$, the equations become $x_2(y_0 - y_1 x_1) = x_1(y_0 - y_2 x_2) = y_1 x_1 - y_2 x_2 = 0$. They define a reducible variety with one irreducible component besides the graph Γ_T isomorphic to $\mathbb{P}^2$ and defined by the equations $x_1 = x_2 = 0$. If we throw it away the graph will be defined by the equations $y_0 - y_1 x_1 = y_0 - y_2 x_2 = y_1 x_1 - y_2 x_2 = 0$. It is easy to see that Γ_T over the affine open set $t_0 \neq 0$ becomes isomorphic to the subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ given by one equation $y_1 x_1 - y_2 x_2 = 0$. This is the definition of the blow-up of the point $[1, 0, 0]$. Replacing p_1 with p_2 and p_2, we find that Γ_T is isomorphic to the blow-up X of the points p_1, p_2, p_3. It is known as a *del Pezzo surface of degree 6*

The complement of the three coordinate lines $t_i = 0$ on the surface X consists of six (-1)-curves E_1, E_2, E_3 and L_1, L_2, L_3 which intersect each other as in the following picture. The images of the curves E_i are the points p_i, and the images of the curves L_i are the coordinate lines $t_i = 0$.

It is clear from the formula for T_2 that the transformation blows down the coordinate line $t_i = 0$

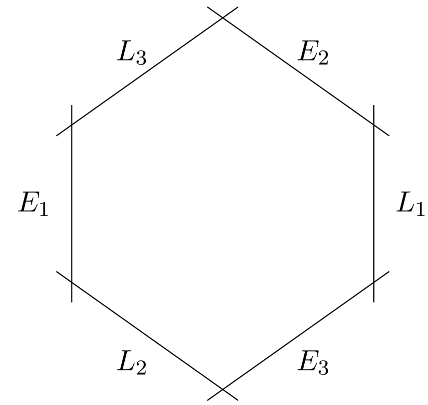


Figure 9.1: del Pezzo surface of degree 6

to the point p_i. The characteristic matrix of T_2 is equal to

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

Note that the matrix is the matrix of the reflection s_{α_1}, where $\alpha_1 = e_0 - e_1 - e_2 - e_3$.

Example 9.3. Recall that the reflections $s_{\mathfrak{e}}$ in $\mathbb{H}^3$ act as the inversion transformations in $\hat{E}^2$ corresponding to the circles defined by the hyperplane of fixed points. This transformation is an example of a (real) Cremona transformation known since the antiquity. Given a circle of radius R, a point $x \in \mathbb{R}^2$ with distance r from the center of the circle is mapped to the point on the same ray at the distance R/r (as in the picture below).

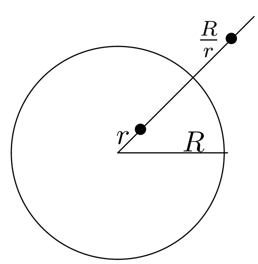


Figure 9.2: Inversion transformation

In the affine plane $\mathbb{C}^2$ the transformation is given by the formula

$$(x, y) \mapsto \Big(\frac{Rx}{x^2+y^2}, \frac{Ry}{x^2+y^2}\Big).$$

In projective coordinates, the transformation is given by the formula

$$(t_0, t_1, t_2) \mapsto (t_1^2 + t_2^2, Rt_1t_0, Rt_2t_0). \tag{9.4}$$

Note that the transformation has three fundamental points $[1, 0, 0]$, $[0, 1, i]$, and $[0, 1, -i]$. It is an involution and transforms lines not passing through the fundamental points to conics (circles in the real affine plane). The lines passing though one of the fundamental points are transformed to lines. The lines passing through the origin $(1, 0, 0)$ are invariant under the transformation. The conic $t_1^2 + t_2^2 - Rt_0^2 = 0$ is the closure of the set of fixed points.

It follows from formula (9.4) that the indeterminacy points of the inversion transformation are three points $p_1 = [1, 0, 0], p_2 = [0, 1, \sqrt{-1}], p_3 = [0, 1, -\sqrt{-1}]$ that lie on the line at infinity $t_0 = 0$. Of course, over reals the only indeterminacy point is the first point. The line $t_0 = 0$ is blown down to the point p_1. The line joining p_1, p_2 (resp. p_1, p_3) is blown down to p_2 (resp. p_3). The characteristic matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

It is the composition of two reflections $s_{\alpha_1} \circ s_{\alpha_2}$, where $\alpha_2 = e_2 - e_3$.

One must be warned that the assigning to a Cremona transformation in the plane its characteristic matrix is not a homomorphism. A subgroup G of $\mathrm{Cr}_n(\Bbbk)$ is called *regularizable* if there exists a rational surface and a birational map $\phi : X \to \mathbb{P}^2$ such that $\phi^{-1} \circ G \circ \phi \subset \mathrm{Aut}(X)$. Here we may assume that X be a basic rational surface. Any finite group is regularizable but not every infinite group is. Let $(e_0, e_1, \ldots, e_n)$ be a geometric basis in $N^1(X)$. Let G be a regularizable subgroup of $\mathrm{Cr}_2(\Bbbk)$ which is realized as a group of biregular automorphisms of X. Then we can realize the characteristic matrix of any $g \in G$ as a matrix of the automorphism $\pi^{-1} \circ g \circ \pi$ in the basis $(e_0, e_1, \ldots, e_n)$. This will give a matrix realization of the natural action homomorphism $G' \to \mathrm{O}(N^1(X))$ and also a homomorphism $G' \to \mathrm{O}(k_X^{\perp}) \cong \mathrm{O}(\mathsf{E}_n)$. The image of this homomorphism is contained in the reflection group $W_{2,3,n-3}$.

Example 9.4. Let X be a rational surface admitting a birational morphism $\pi : X \to \mathbb{P}^2$. We say that X admits a *large group of automorphisms* G if the image of G in $W(\mathsf{E}_n)$ is of finite index. The examples of such surfaces are *Halphen surfaces* and *Coble surfaces* obtained by blowing up 9 base points $p_1, \ldots, p_9$ of a pencil of curves of degree $3m$ of geometric genus 1 that pass through the points $p_1, \ldots, p_9$ with multiplicities equal to m (resp. at ten nodes of an irreducible rational plane curve of degree 6). Under the assumption of generality of such a surface, one can prove that $\mathrm{Aut}(X)$ is isomorphic to a normal subgroup of $W(\mathsf{E}_9)$ (resp. $W(\mathsf{E}_{10})$ with the quotient isomorphic to $(\mathbb{Z}/m\mathbb{Z})^8 \rtimes \mathrm{O}^+(8, \mathbb{F}_2)$ (resp. $\mathrm{O}^+(10, \mathbb{F}_2)$). A theorem from [10] asserts that, if the characteristic of the ground field $\mathrm{char}(\Bbbk)$ is equal to zero, any rational surface with large automorphism group is isomorphic to a general Halphen surface or a general Coble surface. If $\mathrm{char}(\Bbbk) = p > 0$, one has to add one more type of a surface, the blow-up of a general set of $m \geq 9$ points on a cuspidal cubic (a *Harbourne surface*).

There is a similar realization of the Weyl groups $W(\mathsf{E}_{p,q,r})$ for any triple $1 < p \le q \le r$. For this one considers the product $X_{p,q} = (\mathbb{P}^{q-1})^{p-1})$ and an ordered set $\mathcal{P} = (x_1, \dots, x_{q+r})$ of $q + r$ points in $X_{p,q}$. Let $\pi : X^{q+r}(\mathcal{P}) \to X_{p,q}$ be the blow-up of the set of $\mathcal{P}$. The previous case corresponds to $(p, q, r) = (2, 3, N - 3)$. Let $p_i : X_{p,q}(\mathcal{P}) \to X_{p,q} \to \mathbb{P}^{q-1}$ be the ith projection, and $h_i = p_i^*([H])$ is the class in $N^1(X_{p,q}(\mathcal{P}))$ of the pre-image of the class of a hyperplane in $\mathbb{P}^{q-1}$. Let e_i be the class of the exceptional divisor $E_i = \pi^{-1}(x_i)$. Also let $h^i = \pi^*([\ell])$, where $[\ell]$ is the class in $\mathrm{CH}(\mathbb{P}^{q-1})$ of a line in $\mathbb{P}^{q-1}$. Let e^i be the class of a line in E_i. Then, we obtain a basis $(h_1, \dots, h_{p-1}, e_1, \dots, e_{q+r})$ in $\mathrm{Pic}(X_{p,q}(\mathcal{P})$ and the dual basis $(h^1, \dots, h^{p-1}, -e^1, \dots, -e^{q+r}$ in $\mathrm{CH}^{pq-p-q}(X_{p,q})) = \mathrm{Hom}(\mathrm{Pic}(X_{p,q}(\mathcal{P})), \mathbb{Z})$. We have

$$K_{X_{p,q}(\mathcal{P})} = -q(h_1 + \cdots + h_{p-1}) + (pq - p - q)(e_1 + \cdots + e_{q+r}).$$

Let

$$K_X^\vee := -q((h^1 + \cdots + h^{p-1}) + (pq - p - q)(e_1 + \cdots + e_{q+r}$$

Following the definition of the lattice $\mathsf{E}_{p,q,r}$ we define a root basis $(\alpha_1, \dots, \alpha_{p+q+r-1})$ in $(K_X^\vee)^\perp$ and a root basis $(\alpha^1, \dots, \alpha^{p+q+r-1})$ in $K_X^\perp$ which we used in Example 5.8 to define the quadratic lattice $\mathsf{E}_{P,q,r}$. Let $W_{p,q,r}$ be the Weyl group of $\mathsf{E}_{p,q,r}$. It acts on $\mathrm{Pic}(X_{p,q}(\mathcal{P}))$ and on its dual group of 1-cycles.

Assume that $\mathcal{P}$ is a general set of points. More precisely, we assume that no element in the $W_{p,q,r}$-orbit of any root α_i is represented by an effective divisor class (one root is enough). We say in this case that $\mathcal{P}$ is an *unnodal set* of points. Then one can show that, for any $w \in W_{p,q,r}$ the set $(w(e_1), \dots, w(e_{q+r})$ can be represented by divisors $E_1', \dots, E_{q+r}'$ that can be blown down to points $(x_1' \dots, x + q + r') \in X_{p,q}$ (see [20]). Since the blow-down of E_i' is defined only up to a projective automorphism of $X_{p,q}$ and we keep the order of points, we obtain that the set $\mathcal{P}' = (x_1', \dots, x_{q+r}')$ is defined uniquely up to the natural action of $(\mathrm{PGL}_q)^{p-1}$ on the product $X_{p,q} = (\mathbb{P}^{q-1})^{p-1}$.

Let

$$\mathcal{X}_{p,q,r} = X_{p,q}^{q+r}/(\mathrm{PGL}_q)^{p-1}$$

be any birational model of the configuration space of $q + r$ points on $X_{p,q}$ (e.g. we may consider any Geometric Invariant Theory quotient). The previous construction defines a homomorphism

$$\mathrm{cr}_{p,q,r} : W_{p,q,r} \to \mathrm{Bir}(\mathcal{X}_{pqr}) \cong \mathrm{Cr}_{(p-1)(q-1)(r-1)}(\Bbbk).$$

called the *Cremona action* of $W_{p,q,r}$.

Explicitly, it can be defined as follows. For brevity of notation we give this definition only in the case $p = 2$ leaving the general case to the reader (who may consult [20]). The reflections $s_{\alpha_i}, i > 1$, generate the permutation group $\mathfrak{S}_{q+r}$. We let them act on $\mathcal{X}_{p,q,r} = (\mathbb{P}^{q-1})^{q+r}$ by permutation of the factors. The reflection s_{α_1} acts as follows. We choose a representative of $[\mathcal{P}]$ such that the first q points have coordinates $[1, 0, \dots, 0], \dots, [0, \dots, 0, 1]$. Then we consider the Cremona transformation T_{q-1} of $\mathbb{P}^{q-1}$ defined, algebraically, by $(z_1, \dots, z_{q-1}) \mapsto (z_1^{-1}, \dots, z_{q-1}^{-1})$, or, geometrically, by

$$(t_0, \dots, t_q) \mapsto (t_1 \cdots t_q, t_0 t_2 \cdots t_q, \dots, t_0 \cdots t_{q-1}).$$

This is a generalization of the standard quadratic transformation T_2. Then we define the action of the reflection s_{α_1} by sending $[\mathcal{P}]$ to $[\mathcal{P}']$, where $\mathcal{P}' = (x_1, \ldots, x_q, T_{q-1}(x_{q+1}), \ldots, T_{q-1}(x_{p+q})$.

One can interpret the stabilizer subgroup $(W_{p,q,r})_{[\mathcal{P}]}$ of $[\mathcal{P}] \in \mathcal{X}_{p,q,r}$ as follows. An element $w \in (W_{p,q,r})_{[\mathcal{P}]}$ transforms a point set to projectively equivalent point set $\mathcal{P}'$. Let $\Phi : X_{p,q}(\mathcal{P}) \dashrightarrow X_{p,q}(\mathcal{P})$ be a birational map that extends an isomorphism

$$\mathbb{P}^{q-1} \setminus \mathcal{P} \cong X_{p,q}(\mathcal{P}) \setminus (E_1 \cup \ldots \cup E_{q+r}) \to \mathbb{P}^{q-1} \setminus \mathcal{P}' \cong X_{p,q}(\mathcal{P}') \setminus (E_1' \cup \ldots \cup E_{q+r}')$$

The general properties of rational maps of nonsingular projective varieties allow us to extend this birational isomorphism to birational isomorphism of E_i to E_j'. This defines a *pseudo-isomorphism* $\Phi : X_{p,q}(\mathcal{P}) \dashrightarrow X_{p,q}(\mathcal{P}')$. [1] On other hand, since we maya assume that $\mathcal{P} = \mathcal{P}'$, the uniqueness of the blow-up defines an isomorphism $F : X_{p,q}(\mathcal{P}) \to X_{p,q}(\mathcal{P}')$, the composition $F^{-1} \circ \Phi$ is a pseudo-automorphism of $X_{p,q}(\mathcal{P})$.

So, we obtain the following.

Theorem 9.5. *Let $\mathcal{P}$ be an ordered unnodal set of $q + r$ points in $X_{p,q}$ and let $(W_{p,q,r})_{[\mathcal{P}]}$ be the stabilizer subgroup of $W_{p,q,r}$ in the Cremona action of $W_{p,q,r}$ on the configuration space $\mathcal{X}_{p,q,r}$ of $q + r$ points in $X_{p,q}$. Then*

$$(W_{p,q,r})_{[\mathcal{P}]} \subset \mathrm{PsAut}(X_{p,q}(\mathcal{P})),$$

where $\mathrm{PsAut}(X)$ denotes the group of pseudo-automorphisms of a projective algebraic variety X.

Corollary 9.6. *Let*

$$\mathrm{cr}_{2,3,N-3} : W(\mathsf{E}_N) \to \mathrm{Bir}(\mathcal{X}_{2,3,N-3}) \cong \mathrm{Cr}_{2(N-4)}(\Bbbk)$$

be the Cremona action of $W(\mathsf{E}_N)$ on the configuration space of N points in $\mathbb{P}^2$. Then the stabilizer $W(\mathsf{E}_N)_{[P]}$ is isomorphic to a group of automorphisms of the surface X obtained by blowing up an unnodal set of n points in $\mathbb{P}^2$.

This explains in another way why a group of automorphisms of a basic rational surface is isomorphic to a subgroup of $W(\mathsf{E}_N)$.

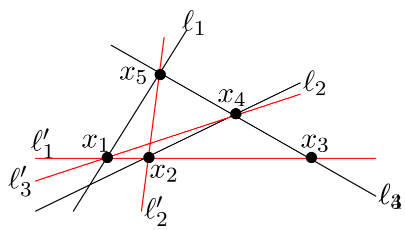
Following A. Coble one says that the closed subvariety Z of $\mathcal{X}_{p,q,r}$ is *special* if Z contains an open subset of configurations of unnodal sets of points such that it is invariant with respect to the Cremona action of $W_{p,q,r}$ and the kernel of the action on Z is a subgroup of finite index. Points sets defining general Halphen or Coble surfaces are examples of special sets of points in the plane. The characterization of Halphen, Coble or Harbourne rational surfaces as surfaces with a large group of automorphisms is equivalent to the classification of special sets of points in the plane

Other examples in higher dimensions discussed in [20].

We already know from the examples of a Coble surface that the Cremona group may contain interesting subgroups isomorphic to discrete subgroups of a hyperbolic space. Here is another example that shows that the universal Coxeter group $\mathsf{UC}(5)$ can be realized as a subgroup of $\mathrm{Cr}_2(\mathbb{C})$.

[1] A pseudo-isomorphism of projective algebraic varieties is a birational map that induces an isomorphism between open subsets with codimension of the complements ≥ 2. For surfaces, a pseudo-isomorphism is an isomorphism.

Example 9.7. Consider a pencil of plane cubic curves spanned by the two cubics which are the union of lines intersecting as in the following picture:



Here one cubic is the union of red lines, and another is the union of blue lines. The pencil defines a rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. We can blow up 9 times, including infinitely near points until we get a commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle\pi} & \searrow{\scriptstyle\tilde{f}} & \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^1 \end{array} .$$

The general member of the regular map $\tilde{f}$ is the proper transform of the general member of the pencil. It is an elliptic curve. We say that the surface X is a *rational elliptic surface*. It can be checked that the fibers corresponding to the cubics equal to the union of three lines are *Kodaira's fibers* of type I_5. A Kodaira fiber of type I_5 is a degeneration of an elliptic curve that is isomorphic to the union of 5 smooth rational curves with self-intersection equal to (-2) whose dual graph (thai is the graph whose vertices correspond to irreducible components and edges to the intersection points taken with multiplicities of the intersections) is a pentagon with simple edges. If we blow up the 5 intersection points in each of these two fibers we obtain a rational surface Y together with a morphism $\phi : Y \to X$. The composition $\pi \circ f : X \to \mathbb{P}^2$ makes it a basic rational surface obtained from $\mathbb{P}^2$ by blowing up 19 points (including infinitely near points). Its Picard number is equal to 20. Let $R_1, \ldots, R_{10}$ be the proper transforms of the components of each fiber of type I_5 on X. The known behavior of the index of intersection of two curves under the blow-up shows that $R_i^2 = -4$. One can show that We know that

$$-2K_Y = [R_1] + \cdots + [R_{10}]$$

but $|-K_Y| = \emptyset$. A rational surface with this property is called a *Coble surface* (see [18]). The formula also tells us that we there exists a double cover $Z \to Y$ branched along the union of curves R_i. The Hurwitz type formula for ramified covers tells us that $K_Z 0$ and Z is a K3 surface. The Picard number of Z is equal to the Picard number of X, hence equal to 20. This is an example of a *singular K3 surface*. This does not mean that Z is singular, but rather that its Picard number is maximal possible for a complex K3 surface. One can compute the Picard lattice and obtain that $N^1(Z)$ is isomorphic to the maximal even sublattice of the lattice $I^{1,19}$. Its orthogonal complement of $H^2(Z, \mathbb{Z})$ is isomorphic to the lattice $\langle 2 \rangle \perp \langle 2 \rangle$. The surface Z is the surface X_4 considered by E. Vinberg in [53]. using the description of the reflection group of $N^1(Z)$ which we presented in

Table 5.1 he shows that $\mathrm{Aut}(Z)$ contains a central involution ι such that

$$\mathrm{Aut}(Z)/(\iota) \cong \mathsf{UC}(5) \rtimes \mathfrak{S}_5.$$

The quotient $Z/(\iota)$ is isomorphic to Y. This gives

$$\mathsf{UC}(5) \rtimes \mathfrak{S}_5 \subset \mathrm{Aut}(Y) \subset \mathrm{Cr}_2(\mathbb{C}).$$

Let $\mathcal{N}$ be the polyhedron in the hyperbolic space $\mathbb{H}^{19}$ associated with $N^1(Z)_\mathbb{R}$ bounded by the hyperplanes H_r, where r is the divisor class of a smooth rational curve on Z. Let $\mathrm{O}^+(N^1(X)$ be the subgroup of $\mathrm{O}(N^1(X))$

The group $\mathsf{UC}(5)$ here coincides with the refThe proof of this result is based on the computation of the Coxeter diagram of the reflection group of the lattice $I^{1,19}$. It is given in the following picture.

Lecture 10

The simplicity of the plane Cremona group

The question whether the group $\mathrm{Cr}_2(\Bbbk)$ is a simple as an abstract group was first raised by F. Enriques in 1894. It took more than a century before this question has been answered negatively by S. Cantat and S. Lamy [11]. In this last lecture we shall discuss a proof of this result.

We have already seen that an infinite subgroup of automorphisms of a basic rational surface acts on the hyperbolic space associated to its numerical lattice. One can define an action of the whole Cremona group $\mathrm{Cr}_2(\Bbbk)$ on a certain infinite-dimensional hyperbolic space $\mathbb{H}^\infty$ introduces by Yuri Manin. This space is defined to be the inductive limit of the Picard groups of basic rational surfaces. More precisely, one considers the category $\mathcal{R}$ whose objects are birational morphisms $\pi : X \to \mathbb{P}^2$ and whose morphisms $f : (X, \pi) \to (X', \pi')$ are regular maps $f : X \to X'$ such that $\pi' \circ f = \pi$. The *bubble space* $\tilde{\mathbb{P}}^2$ is the factor set

$$\Big(\bigcup_{(X \xrightarrow{\pi'} \mathbb{P}^2) \in \mathcal{R}} X \Big) / \sim,$$

where $\sim$ is the following equivalence relation: $p \in X$ is equivalent to $p' \in X'$ if the rational map $\pi'^{-1} \circ \pi : X \dashrightarrow X'$ maps isomorphically an open neighborhood of p to an open neighborhood of p'.

In other words, elements of the bubble space are points on some basic rational surface, and two points are considered to be the same if there exists a birational map from one surface to another which is an isomorphism in an open neighborhood of the points. This gives the needed formalism to define infinitely near points.

A *divisor* on $\tilde{\mathbb{P}}^2$ is a divisor on some basic rational surface $\pi : X \to \mathbb{P}^2$. Two divisors D on $\pi : X \to \mathbb{P}^2$ and D' on $\pi' : X' \to \mathbb{P}^2$ are called linearly equivalent if there exists a morphism $f : X \to X'$ in the category $\mathcal{R}$ such that $f^*(D')$ is linearly equivalent to D on X. The group of linear equivalence classes of divisors is denoted by $N^1(\mathbb{P}^2)^{\mathrm{bb}}$. It is isomorphic to the inductive limit of the Picard groups of basic rational surface.

Let

$$\iota_\pi : \mathrm{Pic}(X) \to N^1(\mathbb{P}^2)^{\mathrm{bb}}$$

be the canonical homomorphism of the Picard group of an object $\pi : X \to \mathbb{P}^2$ of the category $\mathcal{R}$. Let D be a divisor class linearly equivalent to a sum $me_0 + \sum m_i e_i$, where $(e_0, \ldots, e_N)$ is the geometric basis defined by the morphism π. Recall that each e_i is associated to some point p_i in the bubble space. Thus we can identify elements of $N^1(\mathbb{P}^2)^{\mathrm{bb}}$ with formal integral finite sums $me_0 + \sum m_i p_i$, where $p_i \in \tilde{\mathbb{P}}^2$ and e_0 is the class of a line in $\mathbb{P}^2$.

We define the intersection form on $N^1(\mathbb{P}^2)^{\mathrm{bb}}$ extending by additivity the pairing

$$(e_0, e_0) = 0,\ (p, q) = -\delta_{p,q},\ (e_0, p) = 0.$$

It is clear that

$$N^1(\mathbb{P}^2)^{\mathrm{bb}} = N^1(\mathbb{P}^2) \oplus \mathbb{Z}^{\tilde{\mathbb{P}}^2}.$$

The real infinite-dimensional vector space $N^1((\mathbb{P}^2)^{\mathrm{bb}}) \otimes \mathbb{R}$ embeds naturally in the Hilbert Lorentzian space $\mathbb{R}^{1,\infty}$ of infinite sums $d + \sum_{p \in \tilde{\mathbb{P}}^2} a_p p$, where $\sum a_p^2 < \infty$. We can define the associated infinite-dimensional hyperbolic space by extending the vector model of a finite-dimensional hyperbolic space $\mathbb{H}^n$

$$\mathbb{H}^\infty := \{v \in \mathbb{R}^{1,\infty} : (v, v) = 1, (v, e_0) > 0\}.$$

The distance $d(x, y)$ in $\mathbb{H}^\infty$ is defined to beIt is clear that $\mathbb{H}^\infty$ is an inductive limit of $\mathbb{H}^n$.

A divisor on $\tilde{\mathbb{P}}^2$ can be written as a formal sum $de_0 - \sum m_i x_i$, where

$$\cosh\ d(x, y) = (x, y).$$

It is clear nor how to define an abstract $\mathbb{H}^\infty$ by using any metric infinite-dimensional Hilbert space equipped with a quadratic form of signature $(1, \infty)$. For any two points $x, y \in \mathbb{H}^\infty$ there exists a plane in $\mathbb{R}^{1,\infty}$ containing these points. The distance defines a structure of a *metric space* on $\mathbb{H}^\infty$, i.e. it satisfies the triangle inequality. The latter implies that, for any three points x, y, z, one has

$$(y|z)_x = \frac{1}{2}(d(y, x) + d(z, x) - d(y, z)) > 0.$$

A metric space is called *δ-hyperbolic in the sense of Gromov* if, for any four points x, y, z, w,

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta.$$

For example, the Euclidean space is not a δ-hyperbolic for any δ, however a hyperbolic space $\mathbb{H}^n$ is δ-hyperbolic with $\delta = \sqrt{3}$ (see [13], p.11). Since any finite set of points in $\mathbb{H}^\infty$ is contained in some finite-dimensional $\mathbb{H}^n$, we obtain that $\mathbb{H}^\infty$ is $\sqrt{3}$-hyperbolic metric space. For the same reason, any two points can be connected by a geodesic segment. A metric space satisfying this property is called *geodesic*. Thus $\mathbb{H}^\infty$ is a geodesic δ-hyperbolic space. From now on we will be dealing with such space $\mathbb{H}$.

Let us explain how the Cremona group $\mathrm{Cr}_2(\Bbbk)$ acts on $N^1(\mathbb{P}^2)^{\mathrm{bb}}$. Let $X \to \Gamma(\Phi)$ be a resolution of singularities of the graph of T. Composing it with the two projections $\Gamma(\Phi) \to \mathbb{P}^2$, we obtain a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\sigma} \\ \mathbb{P}^2 & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^2 \end{array} \tag{10.1}$$

The first (resp. the second) projection $X \to \mathbb{P}^2$ defines a geometric basis $(e_0, \ldots, e_N)$ (resp. $(e_0', \ldots, e_N')$ and the transformation $e_i \mapsto e_i'$ defined by the characteristic matrix of Φ is an isometry of $N^1(X)$. We extend ϕ^* to an isometry of $N^1(\mathbb{P}^2)^{\mathrm{bb}}$ by making it act identically on the orthogonal complement $N^1(X)^\perp$ in $N^1(\mathbb{P}^2)^{\mathrm{bb}}$. I leave to the reader to check that it defines an action of $\mathrm{Cr}_2(\Bbbk)$ on $N^1(\mathbb{P}^2)^{\mathrm{bb}}$. Note that the introduction of the bubble space and its Picard group makes possible to define the composition of the characteristic matrices of Cremona transformations.

Let G be a group of isometries of $\mathbb{H}$. The paper of Cantat and Lamy provides a criterion when the minimal normal subgroup $\langle\langle g\rangle\rangle$ containing g does not coincide with the whole G.

Let σ be an isometry of $\mathbb{H}$. One defines its *translation length* as the limit

$$L(\sigma) = \lim_{n\to\infty} \frac{d(x, \sigma^n(x))}{n},$$

where x is any point in $\mathbb{H}$. All isometries of $\mathbb{H}$ are divided into three classes: elliptic, parabolic, hyperbolic. An elliptic isometry has a bounded orbit in $\mathbb{H}$ (e.g. a fixed point), a parabolic (resp. hyperbolic) isometry has unbounded orbits and $L(\sigma) = 0$ (resp. $L(\sigma) > 0$). If $\mathbb{H} = \mathbb{H}^n$ this agrees with the earlier definition of elliptic, parabolic and hyperbolic isometries. This follows from our description of such isometries in Lecture 3. A geodesic line invariant under σ is called the *axis* of σ and is denoted by $\mathrm{Ax}(\sigma)$. In $\mathbb{H}^n$ this would be the unique geodesic line joining the two fixed points on σ on the absolute. However, in general, the axis is not unique. Let $\mathrm{Min}(\sigma)$ be the set of points y such that $d(y, \sigma(y)) = L(\sigma)$. It is not empty if and only if σ is hyperbolic. The axis of σ is contained in $\mathrm{Min}(\sigma)$, and σ acts on it by translation of length $L(\sigma)$ along it. Note that one has to assume something on a geodesic δ-hyperbolic space to guarantee that any hyperbolic isometry has an axis (the space must be a CAT(0)-space, whatever it means). It is true for $\mathbb{H}^\infty$.

If A, A' are two subsets of $\mathbb{H}$ and $\alpha \in \mathbb{R}$, we set

$$A \cap_\alpha A' = \{x \in \mathbb{H} : d(x, A) \le \alpha, d(x, A') \le \alpha\}.$$

Let G be a group of isometries of $\mathbb{H}$ and $\sigma \in G$. Let ϵ and B be two positive real numbers, we say that a subset A is (ϵ, B)-*rigid* if $\sigma(A) = A$ as soon as $A \cap_\epsilon \sigma(A) > B$. The set A is ϵ-rigid if there exists $B > 0$ such that A is (ϵ, B)-rigid.

One more definition is in order. A hyperbolic isometry $\sigma \in G$ is called *tight* if

- σ admits a 2θ-rigid axis $\mathrm{Ax}(\sigma)$;
- for all $\sigma' \in G$, if $\sigma'(\mathrm{Ax}(\sigma)) = \mathrm{Ax}(\sigma)$ then $\sigma'\sigma g\sigma^{-1} = \sigma^{\pm 1}$.

Here $\theta = 4\delta$, it is chosen from approximation of finite subsets of $\mathbb{H}$ by trees. We are not going to explain it.

Now, we can state the main result of [11].

Theorem 10.1. *Let G be a group of isometries of $\mathbb{H}$. Suppose $\sigma \in G$ is tight with $(14\theta, B)$-rigid axis $\mathrm{Ax}(g)$. Let n be a positive integer with*

$$nL(\sigma) \geq 20(60\theta + 2B).$$

Then any element $\sigma' \in G \setminus \{1\}$ in the normal subgroup $\langle\langle\sigma\rangle\rangle \subset G$ is conjugate to σ^n unless it is a hyperbolic isometry satisfying $L(\sigma') > nL(\sigma)$. In particular, if $n \geq 2$, the normal subgroup $\langle\langle\sigma^n\rangle\rangle \subset G$ does not contain σ.

Let us explain how this result allows one to prove the simplicity of $\mathrm{Cr}_2(\Bbbk)$. First let us see which Cremona transformations correspond to hyperbolic isometries. Let Φ be a Cremona transformation given by polynomials of degree d which we denote by $\deg(\Phi)$ (not to be confused with the topological degree which is equal to 1). We define the *dynamical degree* of Φ to be

$$\lambda(\Phi) := \overline{\lim}_{n\to\infty} \deg(\Phi^n)^{1/n}.$$

Since the composition of two Cremona transformations is defined by superposition of the corresponding polynomials, we obtain $\deg(\Phi \circ \Phi') \leq \deg(\Phi)\deg(\Phi')$ (the equality happens only if we superposed polynomials are mutually coprime, a rather rare occurrence).

Suppose Φ is realized by an automorphism of some basic rational surface X and hence acts on $N^1(X)$, then the characteristic matrix of Φ^n is equal to A^n, where A is the characteristic matrix of Φ. We know that the degree of Φ is equal to the first entry a_{11} of A. It is obvious that a_{11} is the largest entry of the characteristic matrix. Thus, if we define the norm $||A||$ of a matrix as the largest absolute value of its entries, we obtain

$$||A \cdot B|| \leq ||A||||B||,$$

that is $||A||$ is a matrix norm. The *Gelfand formula* says that

$$\lim_{k\to\infty} ||A^k||^{1/k} = \lambda_1,$$

where λ_1 is the *spectral radius* of A, the largest of the absolute values of eigenvalues of A. Thus, we obtain that the dynamical degree of Φ is equal to the spectral radius of the transformation of Φ on $N^1(X)$. Note that since Φ^* acts by an integral matrix, all eigenvalues of Φ^* are algebraic integers, in particular, the dynamical degree is an algebraic integer. This fact is true in the general case, however the proof of this is rather technical (see [14]).

We have the following characterization of hyperbolic, parabolic or elliptic isometries defined by a Cremona transformation.

Proposition 10.2. *Let Φ be a Cremona transformation of the plane and $\lambda(\Phi)$ be its dynamical degree.Let $\sigma = \Phi^*$ be the corresponding isometry of $N^1(\mathbb{P}^2)^{\mathrm{bb}}$. Then*

- *σ is hyperbolic if $\lambda(\phi) > 1$;*
- *σ is parabolic if $\lambda(\phi) = 1$;*
- *σ is elliptic if $\lambda(\phi) = 0$.*

Moreover, Φ^* is parabolic if and only if Φ preserves a pencil of rational or elliptic curves, the two cases differ by the property that $\deg(\Phi^n)$ grows linearly in the former case and quadratically in the latter case. Also, Φ^* is elliptic if some power of Φ is realized as an element of the connected component of the group of automorphisms of a rational surface.

It is clear that, if Φ is a regularizable Cremona transformation, then Φ^* acts in the hyperbolic space $\mathbb{H}^N$ associated to $N^1(X)_{\mathbb{R}}$ and Φ is hyperbolic, parabolic, or elliptic if and only if the isometry is hyperbolic, parabolic, or elliptic.

Let g be a automorphism of a rational surface X with dynamical degree > 1 (we will call it a *hyperbolic automorphism*) and let g^* be the corresponding hyperbolic isometry of $\mathbb{H}^n, n = \rho(X) - 1$. The axis $\mathrm{Ax}(g^*)$ in $\mathbb{H}^\infty$ is the image of the axis of g^* in $\mathbb{H}^n$. Let V_g be the corresponding plane in $N^1(X)_{\mathbb{R}}$ that contains $\mathrm{Ax}(g^*)$. It is contained in a

Lemma 10.3. *Let g be a hyperbolic automorphism of X. Assume that g^* acts identically on the orthogonal complement $V_g^\perp$ of V_g in $N^1(X)_{\mathbb{R}}$ and the action of $\mathrm{Aut}(X)$ on $N^1(X)$ is faithful. Let h be an automorphism of X such that h^* preserves V_g, then $hgh^{-1} = g^{\pm 1}$.*

Proof. Since h^* preserves V_g and $\mathrm{O}(V_g) \cong \mathrm{O}(1,1) \cong \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$, we obtain that the restrictions of g^* and h^* to V_g either commute or h^* is an involution and $h^*g^*(h^*)^{-1} = (g^*)^{-1}$ on V_g. Since g^* acts identically on $V_g^\perp$, the same is true for the action of g and h on the whole space $N^1(X)_{\mathbb{R}}$. Since $\mathrm{Aut}(X)$ acts faithfully on $N^1(X)_{\mathbb{R}}$, we obtain that the assertion of the lemma. □

Lemma 10.4. *Let h be a birational transformation of a rational surface X. Assume that $V_g \cap N^1(X)$ contains the class η of a hyperplane section of X in some projective embedding. Let $m = \eta^2$ and $\bar{\eta} = \eta/\sqrt{m}$. Suppose*

$$\cosh\, d(\bar{\eta}, h^*(\bar{\eta})) = (\bar{\eta}, h^*(\bar{\eta})) < 1 + \frac{1}{m},$$

Then $h \in \mathrm{Aut}(X)$ and $h^*(\eta) = \eta$.

Proof. Consider the action h^* on $\mathbb{H}^\infty = N^1(\tilde{P}^2)$. We can write $h^*(\bar{\eta}) = \bar{\eta} + r + s$, where $r \in N^1(X)_{\mathbb{R}}$ and $s \in N^1(X)^\perp$. Intersecting both sides with $\bar{\eta}$, we obtain

$$(\bar{\eta}, h^*(\bar{\eta})) = 1 + (r, \bar{\eta}) = 1 + \frac{(r,\eta)}{\sqrt{m}}.$$

Note that $r \in \frac{1}{\sqrt{m}}N^1(X)$, thus $\frac{(r,\eta)}{m} \in \frac{1}{m}\mathbb{Z}$. Using the assumption of the lemma, we get that $(r,\eta) = 0$. This shows that $d(\bar{\eta}, h^*(\bar{\eta})) = 0$, hence $\bar{\eta} = h^*(\bar{\eta})$ and, therefore $\eta = h^*(\eta)$. A birational automorphism preserving the class of a hyperplane section is an automorphism. □

Finally, we can formulate a condition that allows to construct examples of proper normal subgroup of the Cremona group.

Theorem 10.5. *Let g and be a hyperbolic automorphism of a rational surface X. Assume that V_g contains the class η and g^* is the identity on $V_g^\perp \cap N^1(X)$. Then $\mathrm{Ax}(g^*)$ is rigid. Assume, in addition, that if $h \in \mathrm{Aut}(X)$ leaves $\mathrm{Ax}(g^*)$ invariant then $hgh^{-1} = g^{\pm 1}$. Then any birational automorphism of X is an automorphism of X and g^* is a tight element of the image of $\mathrm{Cr}_2(\Bbbk)$ in the group of isometries of $\mathbb{H}^\infty$. Thus some power of g generates a proper normal subgroup of the Cremona group.*

Proof. We keep the notation from the previous lemma. Assume that $\mathrm{Ax}(g^*)$ is not rigid. Then there exists a birational transformation f of X such $d(\bar{\eta}, g^*(\bar{\eta})$ and $d(\bar{\eta}, f^*(\bar{\eta})$ are bounded by $\frac{1}{m}$, and, moreover, $f^*(\mathrm{Ax}(g^*)) \neq \mathrm{Ax}(g^*)$. This follows from Proposition 3.3 from [11]. We omit its proof. Applying the previous Lemma, we obtain that f is an automorphism of X fixing η and $g^*(\eta)$. But then it fixes the ends of the geodesic line $\mathrm{Ax}(g^*)$ and hence fixes $\mathrm{Ax}(g^*)$. This contradiction shows that $\mathrm{Ax}(g^*)$ is rigid.

Assume now that a birational automorphism h of X leaves invariant $\mathrm{Ax}(g^*)$. Since the orbit of η under the cyclic group (g^*) consists of ample divisor classes and they form a dense subset of $\mathrm{Ax}(g^*)$ we see that $h^*(\eta)$ intersects any effective divisor class positively. Hence $h^*(\eta)$ is an ample class, and hence h is an automorphism. The additional assumption implies that g^* is a tight isometry and we conclude by invoking Theorem 10.1. □

Example 10.6. We take X to be a general Coble surface over an algebraically closed field $\Bbbk$. We use a geometric basis $(e_0, e_1, \ldots, e_{10})$ on X arising from the blowing up $X \to \mathbb{P}^2$ of 10 double points $p_1, \ldots, p_{10}$ of an irreducible sextic C. Consider the curves D_1 (resp. D_2) of degree 6 passing through the points $p_1, \ldots, p_{10}$ with double points at $p_1, \ldots, p_8$ (resp. $p_1, \ldots, p_6, p_9, p_{10}$). The divisors classes of these curves are

$$[D_1] = -2K_X + e_9 + e_{10},\ [D_2] = -2K_X + e_7 + e_8.$$

We have $D_1^2 = D_2^2 = 2, D_1 \cdot D_2 = 4$. Let M be the sublattice of $N^1(X)$ spanned by $[D_1], [D_2]$. Its quadratic form is defined by the matrix $\left(\begin{smallmatrix} 2 & 4 \\ 4 & 2 \end{smallmatrix}\right)$. Consider the isometry σ_0 of M defined by the matrix $\left(\begin{smallmatrix} 4 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the basis $([D_1], [D_2])$.[1] Its eigenvalues are $2 \pm \sqrt{3}$. So the spectral radius is $\lambda_1 = 2 \pm \sqrt{3} > 1$. The isometry σ_0^2 is defined by the matrix $\left(\begin{smallmatrix} 15 & 4 \\ -4 & -1 \end{smallmatrix}\right)$. The discriminant group of M is generated by $e^* = \frac{1}{6}[D_1] + \frac{1}{3}[D_2]$. It is a finite group of order 12. One checks immediately that $\sigma_0^2(e^*) - e^* \in M$. This implies that σ_0^2 extends to an isometry σ on $N^1(X)$ that acts identically on the orthogonal complement of M. It also acts identically on $N^1(X)$ modulo $2N^1(X)$ (i.e.

[1] Beware that there is a typo in the definition of this isometry in [11], p. 83.

$\frac{1}{2}(\sigma(v)-v)$ exists in $N^1(X)$). It is known that $\mathrm{Aut}(X)^*$ for a general Coble surface is isomorphic to $\mathrm{Aut}(X)$ and contains the group of such isometries (see a proof in [10]). Thus $\sigma = g^*$ for some automorphism of X. The dynamical degree of g is equal to $(2+\sqrt{3})^2 = 4+2\sqrt{3} > 1$. Note that V_g contains an ample divisor class, any positive integral combination of D_1 and D_2 will do. It is obvious that any hyperbolic h leaving $\mathrm{Ax}(g^*)$ invariant must commute with g^* (because $\dim V_g = 2$, its action is determined uniquely by its action on $\mathrm{Ax}(g)$). Thus, applying lemma 10.3, we see that all conditions of Theorem 10.5 are satisfied. Thus some power of g generates a proper normal subgroup of $\mathrm{Cr}_2(\Bbbk)$.

Let us find the Cremona transformation corresponding to the automorphism g. Write $e_0 = a[D_1]+b[D_2]+x$, where $x \in M^\perp$. Intersecting both sides with $[D_1]$ and $D_2]$, we obtain $a=b=1$. This gives

$$g^*(e_0) = 19[D_1]-5[D_2]+x = 19[D_1]-5[D_2]+e_0-([D_1]+[D_2]) = 18[D_1]-6[D_2]+e_0$$

$$= -24K_X+18(e_9+e_{10})-6(e_7+e_8)+e_0 = 73e_0-24(e_1+\cdots+e_6)-30(e_7+e_8)-6(e_9+e_{10}).$$

This shows that g is given by the linear system of plane curves of degree 73 with points of multiplicity 24 at $p_1,\ldots,p_6$, points of multiplicity 30 at p_7,p_8 and points of multiplicity 6 at p_9,p_{10}.

Example 10.7. Let E be an elliptic curve with complex multiplication by the ring $\mathfrak{o}$ of Gaussian. We know from Example 6.10 that the group $\mathrm{SL}_2(\mathbb{Z}[i])$ acts on the abelian surface $A = E\times E$. Let $\iota \in \mathrm{Aut}(E\times E)$ be defined by $(z_1,z_2)\mapsto(iz_1,iz_2)$. One can show that the quotient of $E\times E$ by the cyclic group generated by ι is a rational surface (it is the quotient of the Kummer surface of A by an involution with no isolated fixed points). Obviously, the group $\mathrm{PSL}_2(\mathbb{Z}[i]) = \mathrm{SL}_2(\mathbb{Z}[i])/iI_2$ acts on the quotient, and hence embeds in $\mathrm{Cr}_2(\mathbb{C})$. Cantat and Lamy show that, for any element g of $\mathrm{PSL}_2(\mathbb{Z}[i])$ represented by a matrix with trace >2, there exists some power of g that generates a proper normal subgroup $\mathrm{Cr}_2(\mathbb{C})$.

Finally note, that the existence of an element g whose power generates a proper normal subgroup is a general phenomenon. Cantat and Lamy prove, for example, that a general quadratic transformation has such property.

Finally we refer the reader to [36] and [49] for some improvements of the main results of Cantat's and Lamy's paper (examples of proper normal subgroups in over non-algebraically closed field and effectiveness of the powers of an element generating such a subgroup).

Bibliography

[1] D. Alekseevskii, E. Vinberg, A. Solodovnikov, *Geometry of spaces of constant curvature*. Geometry, II, 1–138, Encyclopaedia Math. Sci., 29, Springer, Berlin, 1993.

[2] B. Apanasov, *Conformal geometry of discrete groups and manifolds*. de Gruyter Expositions in Mathematics, 32. Walter de Gruyter, Berlin, 2000.

[3] A. Baragar, *Orbits of curves on certain K3 surfaces*. Compositio Math. **137** (2003), 115–134.

[4] A. Baragar, *The ample cone for a K3 surface*. Canad. J. Math. **63** (2011), 481–499.

[5] M. Belolipetsky, J. Mcleod, *Reflective and quasi-reflective Bianchi groups*. Transform. Groups **18** (2013), 971–994.

[6] Y. Benoist, P. de la Harpe, *Adhérence de Zariski des groupes de Coxeter*. Compos. Math. **140** (2004), 1357–1366.

[7] D. Boyd, *A new class of infinite sphere packings*. Pacific J. Math. **50** (1974), 383–398

[8] D. Boyd, *The sequence of radii in an Apollonian packing*, Math. Comp. **39** (1982) 249–254.

[9] J. Bourgain, E. Fuchs, *A proof of the positive density conjecture for integer Apollonian circle packings*. J. Amer. Math. Soc. **24** (2011), 945–967.

[10] S. Cantat, I. Dolgachev, *Rational surfaces with a large group of automorphisms*. J. Amer. Math. Soc. **25** (2012), 863–905.

[11] S. Cantat, S. Lamy, *Normal subgroups in the Cremona group. With an appendix by Yves de Cornulier*. Acta Math. **210** (2013), 31–94

[12] H. Chen, J.-Ph. Labbé, Lorentzian Coxeter systems and Boyd–Maxwell ball packings. arXiv:1310.8608, math.GR.

[13] M. Coornaert, T. Delzant, A. Papadopoulos, *Géomtrie et théorie des groupes*. Lecture Notes in Mathematics, **1441**. Springer-Verlag, Berlin, 1990.

[14] J. Diller, C. Favre, *Dynamics of bimeromorphic maps of surfaces*. Amer. J. Math. **123** (2001), 1135–1169.

[15] S. Dani, *On invariant measures, minimal sets and a lemma of Margulis*. Invent. Math. **51** (1979), 239–260.

[16] I. Dolgachev, *Automorphisms of Enriques surfaces*, Invent. Math. **76** (1984), 63–177.

[17] I. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*. Algebraic geometry, **4** J. Math. Sci. **81** (1996), 2599–2630.

[18] I. Dolgachev, De-Qi Zhang,*Coble rational surfaces*. Amer. J. Math. **123** (2001), 79–114.

[19] I. Dolgachev, *Reflection groups in algebraic geometry*. Bull. Amer. Math. Soc. (N.S.) **45** (2008), 1–60.

[20] I. Dolgachev, *Cremona special sets of points in products of projective spaces*. Complex and differential geometry, 115–134, Springer Proc. Math., 8, Springer, Heidelberg, 2011.

[21] I. Dolgachev, *A brief introduction to Enriques surfaces*, Development of moduli theory, Advanced Studies in Pure Mathematics, Math. Soc. Japan, to appear.

[22] I. Dolgachev, B. Howard, *Configuration spaces of complex and real spheres*, Recent Advances in Algebraic Geometry, ed. C. Hacon, M. Mustata and M. Popa, London Math. Soc. Lect. Notes, Cambridge Univ. Press, 2015, 156–179.

[23] W. Duke, Z. Rudnick, P. Sarnak, P., *Density of integer points on affine homogeneous varieties*. Duke Math. J. **71** (1993), 143–179.

[24] A. Eskin, C. McMullen, *Mixing, counting, and equidistribution in Lie groups*. Duke Math. J. **71** (1993), 181–209.

[25] W. Fulton, *Intersection theory*, Springer-Verlag. 1984.

[26] R. Graham, J. Lagarias, C. Mallows, A. Wilks, C. Yan, *Apollonian circle packings: number theory*. J. Number Theory **100** (2003), 1–45.

[27] R. Graham, J. Lagarias, C. Mallows, A. Wilks, C. Yan, *Apollonian circle packings: geometry and group theory. I. The Apollonian group*. Discrete Comput. Geom. **34** (2005), 547–585.

[28] R. Graham, J. Lagarias, C. Mallows, A. Wilks, C. Yan, *Apollonian circle packings: geometry and group theory. II. Super-Apollonian group and integral packings*. Discrete Comput. Geom. **35** (2006), 1–36.

[29] R. Graham, J. Lagarias, C. Mallows, A. Wilks, C. Yan, *Apollonian circle packings: geometry and group theory. III. Higher dimensions*. Discrete Comput. Geom. **35** (2006), 37–72.

[30] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[31] M. Kapovich, L. Potyagailo, E. Vinberg, *Noncoherence of some lattices in Isom(H^n)*. The Zieschang Gedenkschrift, 335–351, Geom. Topol. Monogr., 14, Geom. Topol. Publ., Coventry, 2008.

[32] S. Kleiman, *The standard conjectures*. Motives (Seattle, WA, 1991), 320, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[33] A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*. With an appendix by H.Oh and N. Shah. J. Amer. Math. Soc. 24 (2011), 603–648.

[34] P. Lax, R. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*. J. Funct. Anal. **46** (1982), 280–350.

[35] J. Lewis, *A survey of the Hodge conjecture*. Introductory lectures in transcendental algebraic geometry. Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1991.

[36] A. Lonjou, *Non simplicité du groupe de Cremona a sur tout corps*, arXiv:1503.03731v1[math.AG], 12 Mar 2015.

[37] C. McMullen, *Hausdorff dimension and conformal dynamics. III. Computation of dimension*. Amer. J. Math. **120** (1998), 691–721.

[38] G. Maxwell, *Sphere packings and hyperbolic reflection groups*. J. Algebra **79** (1982), 78-97.

[39] G. Maxwell, *Wythoff's construction for Coxeter groups*. J. Algebra **123** (1989), 351–377.

[40] A. Mohammadi, H. Oh, *Matrix coefficients, Counting and Primes for orbits of geometrically finite groups*. arXiv:1208.4139 math.NT.

[41] V. Nikulin, *Integral quadratic forms and some of its geometric applications*, Izv. Akad. Nauk SSSR, Ser. Math. **43** (1979), 111–177.

[42] V. Nikulin, *K3 surfaces with a finite group of automorphisms and a Picard group of rank three*. Algebraic geometry and its applications. Trudy Mat. Inst. Steklov. **165** (1984), 119–142.

[43] H. Oh, Harmonic analysis, Ergodic theory and Counting for thin groups, MSRI publication Vol. 61 (2014) "Thin groups and Superstrong approximation" edited by E. Breuilliard and H. Oh,2014.

[44] H. Oh, N. Shah, *Equidistribution and counting for orbits of geometrically finite hyperbolic groups*. J. Amer. Math. Soc. **26** (2013), 511–562.

[45] J. Ratcliffe, *Foundations of hyperbolic manifolds*. Second edition. Graduate Texts in Mathematics, 149. Springer, New York, 2006.

[46] T. Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. (N.S.) **95** (2003), vi+96 pp.

[47] P. Sarnak, *Integral Apollonian packings*. Amer. Math. Monthly **118** (2011), 291–306.

[48] J.-P. Serre, *Cours de Arithmétique*, Pres. Univ. de France, Paris, 1970.

[49] N. Shepherd-Barron, *Some effectivity questions for plane Cremona group of transformation*, arXiv:1311.6608v2[math.AG] 24 Feb 2014.

[50] D. Sullivan, *Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*. Acta Math. **153** (1984), 259–277.

[51] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*. Inst. Hautes Études Sci. Publ. Math. **50** (1979), 171–202.

[52] P. Thomas, D. Dhar, *The Hausdorff dimension of the Apollonian packing of circles*. J. Phys. A **27** (1994), no. 7, 2257–2268.

[53] E. Vinberg, *The two most algebraic K3-surfaces*, Math Ann. **26** (1983), 1–21.

[54] E. Vinberg, O. Shvartsman, *Discrete groups of motions of spaces of constant curvature*. Geometry, II, 139–248, Encyclopaedia Math. Sci., 29, Springer, Berlin, 1993.

[55] E. Vinberg, *Classification of 2-reflective hyperbolic lattices of rank 4*. Tr. Mosk. Mat. Obs. **68** (2007), 44–76; translation in Trans. Moscow Math. Soc. 2007, 39–66.

Index